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Bishop Frames of Salkowski Curves in E^3

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Abstract
In this study, alternative, type-1 Bishop, type-2 Bishop and N-Bishop frames of Salkowski curves in E^3 are calculated. Moreover, curvatures, Darboux and pol vectors of these frames are found. Also, relationships between the Bishop frames, Darboux vectors and pole vectors are given.

1. Introduction

By defining a moving frame at every point on any curve, it is possible to examine the characteristic and kinematic properties of the curve. Therefore, defining a new and useful frame on any curve is always a current and interesting field of study, especially for geometers. One of the best known of the frames defined on any curve is the Frenet frame. This frame consists of three linearly independent orthonormal vectors obtained from derivatives of the curve, [1]. Alternative frame is produced from the elements of the Frenet frame, [2]. Another well-known frame, the Bishop frame, is a relatively parallel frame obtained by rotating the Frenet frame around the T vector by an angle, [3]. This frame is known to be more advantageous than the Frenet frame, which works well even when the second derivative of the curve has vanishing. Therefore, it is a subject that not only geometry but also biology and computer graphics, [4,5]. Based on this frame, type-2 Bishop frame was introduced in [6] and N-Bishop frame was introduced in [7]. The Type-2 Bishop frame is obtained by rotating the Frenet frame of the curve around the B vector by a certain angle, while the N-Bishop frame is obtained by rotating the alternative frame of the curve around N by a certain angle. Some other studies on these frames are [8-21]. On the other hand, Salkowski curves in E^3 are slant helix type curves introduced by Salkowski, [22]. The Frenet vectors and curvatures of these curves with constant curvature but not constant torsion were found by Monterde, [23]. The Darboux and pole vectors belonging to the Frenet frame and modified frames of Salkowski curves E^3 are studied in [24]. Other some studies on Salkowski curves in E^3 can be looked at from [25-28]. In this study, alternative, type-1, type-2 and N-Bishop frames of Salkowski curves are calculated and curvatures, Darboux and pole vectors belonging to the frames are investigated. Besides, the relations between these elements are given. The aim of this study is to define new frames on Salkowski curves. Although the Frenet frame of the Salkowski curve works smoothly, the literature richness of the curve has been increased with new frames defined on it.

2. Material and Method

Frenet frame $\{T,N,B\}$ of any non-unit speed (with an arbitrary parameter t) regular curve ψ in E^3 is

$$T=\frac{\psi'}{\|\psi'\|},\qquad N=B\wedge T,\qquad B=\frac{\psi'\wedge\psi''}{\|\psi'\wedge\psi''\|}$$

and curvature $\aleph$ and torsion $\Im$ of ψ are

$$\aleph=\frac{\|\psi'\wedge\psi''\|}{\|\psi'\|^3},\qquad \Im=\frac{\langle\psi',\psi'',\psi'''\rangle}{\|\psi'\wedge\psi''\|^2}$$

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[1]. Darboux vector and pole vector belonging to the Frenet frame of ψ are

$$\begin{cases} \mathcal{F} = N \wedge N' = \|\psi'\|\left(\Im T + \aleph B\right), \\ \mathcal{C} = \dfrac{\Im}{\sqrt{\aleph^2+\Im^2}}T + \dfrac{\aleph}{\sqrt{\aleph^2+\Im^2}}B, \end{cases}$$

where

$$T' = \mathcal{F} \wedge T, \quad N' = \mathcal{F} \wedge N, \quad B' = \mathcal{F} \wedge B.$$

Type-1 Bishop frame $\{T, N_1, B_1\}$ of any non-unit speed regular curve ψ in E^3 is [3]

$$\begin{cases} T = \dfrac{\psi'}{\|\psi'\|}, \\ N_1 = \cos\Theta N - \sin\Theta B, \\ B_1 = T \wedge N_1 = \sin\Theta N + \cos\Theta B, \\ \Theta = \int \|\psi'\| \Im \, dt, \end{cases} \tag{1}$$

curvature $\aleph_1$ and torsion $\Im_1$ of ψ are

$$\aleph_1 = \aleph \cos\Theta, \quad \Im_1 = \aleph \sin\Theta. \tag{2}$$

The matrix representation of type-1 Bishop derivative formulas of ψ is

$$\begin{bmatrix} T' \\ N_1' \\ B_1' \end{bmatrix} = \begin{bmatrix} 0 & \|\psi'\|\aleph_1 & \|\psi'\|\Im_1 \\ -\|\psi'\|\aleph_1 & 0 & 0 \\ -\|\psi'\|\Im_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ B_1 \end{bmatrix}. \tag{3}$$

Darboux vector belonging to the type-1 Bishop frame of ψ is [8]

$$\mathcal{F}_1 = T \wedge T' = \|\psi'\|\left(-\Im_1 N_1 + \aleph_1 B_1\right), \tag{4}$$

where

$$T' = \mathcal{F}_1 \wedge T, \quad N_1' = \mathcal{F}_1 \wedge N_1, \quad B_1' = \mathcal{F}_1 \wedge B_1.$$

Type-2 Bishop frame $\{N_2, B_2, B\}$ of any non-unit speed regular curve ψ in E^3 is [6]

$$\begin{cases} N_2 = \sin\Phi T + \cos\Phi N, \\ B_2 = B \wedge N_2 = -\cos\Phi T + \sin\Phi N, \\ B = T \wedge N, \\ \Phi = \int \|\psi'\| \aleph \, dt, \end{cases} \tag{5}$$

curvature $\aleph_2$ and torsion $\Im_2$ of ψ are

$$\aleph_2 = -\Im \cos\Phi, \quad \Im_2 = -\Im \sin\Phi. \tag{6}$$

The matrix representation of type-2 Bishop derivative formulas of ψ is

$$\begin{bmatrix} N_2' \\ B_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\|\psi'\|\aleph_2 \\ 0 & 0 & -\|\psi'\|\Im_2 \\ \|\psi'\|\aleph_2 & \|\psi'\|\Im_2 & 0 \end{bmatrix} \begin{bmatrix} N_2 \\ B_2 \\ B \end{bmatrix}. \tag{7}$$

Darboux vector belonging to the type-2 Bishop frame of ψ is [17]

$$\mathcal{F}_2 = B \wedge B' = \|\psi'\|\left(-\Im_2 N_2 + \aleph_2 B_2\right), \tag{8}$$

where

$$N_2' = \mathcal{F}_2 \wedge N_2, \quad B_2' = \mathcal{F}_2 \wedge B_2, \quad B' = \mathcal{F}_2 \wedge B.$$

Alternative frame $\{N, C, W\}$ of any non-unit speed regular curve ψ in E^3 is [2]

$$\begin{cases} N = B \wedge T, \\ C = \dfrac{N'}{\|N'\|} = -\dfrac{\aleph}{\sqrt{\aleph^2+\Im^2}}T + \dfrac{\Im}{\sqrt{\aleph^2+\Im^2}}B, \\ W = N \wedge C = \dfrac{\Im}{\sqrt{\aleph^2+\Im^2}}T + \dfrac{\aleph}{\sqrt{\aleph^2+\Im^2}}B, \end{cases} \tag{9}$$

curvature F and torsion G are

$$F = \sqrt{\aleph^2+\Im^2}, \quad G = \frac{\aleph\Im' - \aleph'\Im}{F^2}. \tag{10}$$

The matrix representation of alternative derivative formulas of ψ is

$$\begin{bmatrix} N' \\ C' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & \|\psi'\|F & 0 \\ -\|\psi'\|F & 0 & G \\ 0 & -G & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix}. \tag{11}$$

Darboux vector belonging to the alternative frame of ψ is [19]

$$\overline{\mathcal{F}} = C \wedge C' = GN + \|\psi'\|FW, \tag{12}$$

where

$$N' = \overline{\mathcal{F}} \wedge N, \quad C' = \overline{\mathcal{F}} \wedge C, \quad W' = \overline{\mathcal{F}} \wedge W.$$

N-Bishop frame $\{N, N_3, B_3\}$ of any non-unit speed regular curve ψ in E^3 is [7]

$$\begin{cases} N = B \wedge T \\ N_3 = \cos\Omega C - \sin\Omega W, \\ B_3 = N \wedge N_3 = \sin\Omega C + \cos\Omega W, \\ \Omega = \int G\, dt, \end{cases} \tag{13}$$

curvature $\aleph_3$ and torsion $\Im_3$ of ψ are

$$\aleph_3 = F\cos\Omega, \qquad \Im_3 = F\sin\Omega. \tag{14}$$

The matrix representation of N-Bishop derivative formulas of ψ is

$$\begin{bmatrix} N' \\ N_3' \\ B_3' \end{bmatrix} = \begin{bmatrix} 0 & \|\psi'\|\aleph_3 & \|\psi'\|\Im_3 \\ -\|\psi'\|\aleph_3 & 0 & 0 \\ -\|\psi'\|\Im_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix}. \tag{15}$$

Darboux vector belonging to the N-Bishop of ψ is

$$\mathcal{F}_3 = N \wedge N' = \|\psi'\|\left(-\Im_3 N_3 + \aleph_3 B_3\right), \tag{16}$$

[17], where

$$N' = \mathcal{F}_3 \wedge N, \qquad N_3' = \mathcal{F}_2 \wedge N_3, \qquad B_3' = \mathcal{F}_2 \wedge B_3.$$

Definition 2.1. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and

$$n = \frac{m}{\sqrt{m^2+1}},$$

$$\begin{aligned} \psi_m = \frac{n}{4m}\Bigg(& \frac{n-1}{1+2n}\sin\left((1+2n)t\right) \\ & -\frac{1+n}{1-2n}\sin\left((1-2n)t\right) - 2\sin t, \\ & \frac{1-n}{1+2n}\cos\left((1+2n)t\right) \\ & +\frac{1+n}{1-2n}\cos\left((1-2n)t\right) + 2\cos t, \frac{1}{m}\cos(2nt)\Bigg) \end{aligned}$$

is the parametric equation of Salkowski curves in E^3, Figure 1, [22]. The curves are regular in the interval of $\left]-\frac{\pi}{2n}, \frac{\pi}{2n}\right[$ and

$$\left\|\psi_m'\right\| = \frac{n}{m}\cos(nt). \tag{17}$$

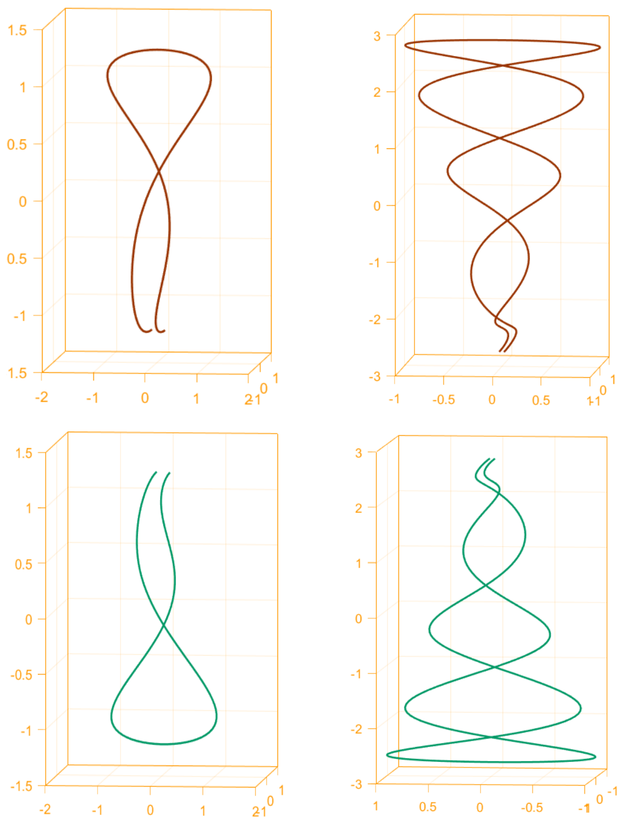

Figure 1. Salkowski curves for $m = \frac{1}{5}, \frac{1}{11}, -\frac{1}{5} - \frac{1}{11}$, respectively

Frenet frame $\{T, N, B\}$ of ψ_m is [23]

$$\begin{cases} T = \Big(-\cos t\cos(nt) - n\sin t\sin(nt), \\ \quad -\sin t\cos(nt) + n\cos t\sin(nt), \\ \quad -\frac{n}{m}\sin(nt)\Big), \\ N = \left(\frac{n}{m}\sin t, -\frac{n}{m}\cos t, -n\right), \\ B = \Big(-\cos t\sin(nt) + n\sin t\cos(nt), \\ \quad -\sin t\sin(nt) - n\cos t\cos(nt), \\ \quad \frac{n}{m}\cos(nt)\Big), \end{cases} \tag{18}$$

curvature $\aleph$ and torsion $\Im$ of ψ_m are [23, 24]

$$\aleph = 1, \qquad \Im = -\tan(nt). \tag{19}$$

Darboux vector $\mathcal{F}$ and pole vector $\mathcal{C}$ (or unit vector in the direction of Darboux vector) belonging to the Frenet frame of ψ_m are [24]

$$\begin{cases} \mathcal{F} = \left(\dfrac{n^2}{m} \sin t, -\dfrac{n^2}{m} \cos t, \dfrac{n^2}{m^2} \right), \\ \mathcal{C} = \left(n \sin t, -n \cos t, \dfrac{n}{m} \right). \end{cases} \tag{20}$$

3. Bishop Frames of Salkowski Curves in E^3

In this section, type-1 Bishop, type-2 Bishop, alternative and N-Bishop frames of Salkowski curves ψ_m in E^3 will be examined, respectively. Besides, Darboux and pole vectors belonging to these frames of ψ_m will be computed.

3.1. Type-1 Bishop Frame of Salkowski Curves in E^3

Theorem 3.1. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, type-1 Bishop frame $\{T, N_1, B_1\}$ of ψ_m obtained by rotating Frenet frame of ψ_m around T by an angle Θ is as follows:

$$\begin{cases} T = \Big(-\cos t \cos(nt) - n \sin t \sin(nt), \\ \quad -\sin t \cos(nt) + n \cos t \sin(nt), \\ \quad -\dfrac{n}{m} \sin(nt) \Big), \\ N_1 = \Big(\dfrac{n}{m} \cos\Theta \sin t + \sin\Theta \cos t \sin(nt) \\ \quad - n \sin\Theta \sin t \cos(nt), \\ \quad -\dfrac{n}{m} \cos\Theta \cos t + \sin\Theta \sin t \sin(nt) \\ \quad + n \sin\Theta \cos t \cos(nt), \\ \quad -n\cos\Theta - \dfrac{n}{m} \sin\Theta \cos(nt) \Big), \\ B_1 = \Big(\dfrac{n}{m} \sin\Theta \sin t - \cos\Theta \cos t \sin(nt) \\ \quad + n \cos\Theta \sin t \cos(nt), \\ \quad -\dfrac{n}{m} \sin\Theta \cos t - \cos\Theta \sin t \sin(nt) \\ \quad - n \cos\Theta \cos t \cos(nt), \\ \quad -n \sin\Theta + \dfrac{n}{m} \cos\Theta \cos(nt) \Big). \end{cases} \tag{21}$$

Proof: The proof is obvious that from (1) and (18).

Corollary 3.1. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, type-1 Bishop frame of ψ_m is obtained by rotating Frenet frame of ψ_m around T by an angle Θ:

$$\Theta = \frac{1}{m} \cos(nt) + c_1, \quad c_1 \in R. \tag{22}$$

Proof: From (1), (17) and (19),

$$\begin{aligned} \Theta &= \int \left\| \psi_m{}' \right\| \mathfrak{I} \, dt = -\frac{n}{m} \int \sin(nt) dt \\ &= \frac{1}{m} \cos(nt) + c_1, \quad c_1 \in R \end{aligned}$$

is obtained.

Corollary 3.2. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, there is the following matrix relation between of type-1 Bishop frame $\{T, N_1, B_1\}$ and Frenet frame $\{T, N, B\}$ of ψ_m:

$$\begin{bmatrix} T \\ N_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{1}{m}\cos(nt) + c_1\right) & -\sin\left(\frac{1}{m}\cos(nt) + c_1\right) \\ 0 & \sin\left(\frac{1}{m}\cos(nt) + c_1\right) & \cos\left(\frac{1}{m}\cos(nt) + c_1\right) \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Proof: The proof is obvious that from (1) and (22).

Theorem 3.2. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, curvature $\aleph_1$ and torsion $\mathfrak{I}_1$ of type-1 Bishop frame $\{T, N_1, B_1\}$ of ψ_m obtained by rotating Frenet frame of ψ_m around T by an angle Θ are as follows:

$$\begin{cases} \aleph_1 = \cos\Theta = \cos\left(\dfrac{1}{m} \cos(nt) + c_1 \right), \\ \mathfrak{I}_1 = \sin\Theta = \sin\left(\dfrac{1}{m} \cos(nt) + c_1 \right). \end{cases} \tag{23}$$

Proof: The proof is obvious that from (2), (19) and (22).

Corollary 3.3. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, derivative vectors of Bishop frame $\{T, N_1, B_1\}$ of ψ_m are as follows:

$$\begin{cases} T' = \frac{n^2}{m^2}\cos(nt)(\sin t, -\cos t, -m), \\ N_1' = \Big(\frac{n}{m}\cos\Theta\cos t\cos^2(nt) \\ \quad + \frac{n^2}{m}\cos\Theta\sin t\cos(nt)\sin(nt), \\ \quad \frac{n}{m}\cos\Theta\sin t\cos^2(nt) \\ \quad - \frac{n^2}{m}\cos\Theta\cos t\cos(nt)\sin(nt), \\ \quad \frac{n^2}{m^2}\cos\Theta\cos(nt)\sin(nt)\Big), \\ B_1' = \Big(\frac{n}{m}\sin\Theta\cos t\cos^2(nt) \\ \quad + \frac{n^2}{m}\sin\Theta\sin t\cos(nt)\sin(nt), \\ \quad \frac{n}{m}\sin\Theta\sin t\cos^2(nt) \\ \quad - \frac{n^2}{m}\sin\Theta\cos t\cos(nt)\sin(nt), \\ \quad \frac{n^2}{m^2}\sin\Theta\cos(nt)\sin(nt)\Big). \end{cases} \tag{24}$$

Proof: From (3), (17), (21) and (23), the vectors

$$N_1' = -\left\|\psi_m'\right\|\aleph_1 T,$$

$$B_1' = -\left\|\psi_m'\right\|\Im_1 T,$$

$$T' = \left\|\psi_m'\right\|\aleph_1 N_1 + \left\|\psi_m'\right\|\Im_1 B_1$$

are obtained as in (24). These vectors can be obtained in the same way by taking the derivatives of the vectors in (22).

Theorem 3.3. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, the matrix representation of type-1 Bishop derivative formulas of ψ_m:

$$\begin{bmatrix} T' \\ N_1' \\ B_1' \end{bmatrix} = \frac{n}{m}\cos(nt)\begin{bmatrix} 0 & \cos\Theta & \sin\Theta \\ -\cos\Theta & 0 & 0 \\ -\sin\Theta & 0 & 0 \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ B_1 \end{bmatrix}.$$

Proof: The proof is obvious that from (3), (17) and (23). Also, it is also obtained by comparing expressions (21) and (24).

Theorem 3.4. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, Darboux vector $\mathcal{F}_1$ belonging to the type-1 Bishop frame of ψ_m is as follows:

$$\begin{aligned} \mathcal{F}_1 = \Big(& -\frac{n}{m}\cos t\cos(nt)\sin(nt) + \frac{n^2}{m}\sin t\cos^2(nt), \\ & -\frac{n}{m}\sin t\cos(nt)\sin(nt) - \frac{n^2}{m}\cos t\cos^2(nt) \\ & \frac{n^2}{m^2}\cos^2(nt)\Big). \end{aligned} \tag{25}$$

Proof: If (17), (21) and (23) are substituted in (4), (25) is obtained.

Theorem 3.5. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, pole vector $\mathcal{C}_1$ belonging to the type-1 Bishop frame of ψ_m is as follows:

$$\begin{aligned} \mathcal{C}_1 = \Big(& -\cos t\sin(nt) + n\sin t\cos(nt), \\ & -\sin t\sin(nt) - n\cos t\cos(nt), \\ & \frac{n}{m}\cos(nt)\Big). \end{aligned} \tag{26}$$

Proof: From (4) and (23), pole vector (unit vector in the direction of Darboux vector) belonging to the type-1 Bishop frame of ψ_m is

$$\begin{aligned} \mathcal{C}_1 = \frac{\mathcal{F}_1}{\left\|\mathcal{F}_1\right\|} &= -\frac{\Im_1}{\sqrt{\aleph_1^2 + \Im_1^2}}N_1 + \frac{\aleph_1}{\sqrt{\aleph_1^2 + \Im_1^2}}B_1 \\ &= -\Im_1 N_1 + \aleph_1 B_1 \\ &= -\sin\Theta N_1 + \cos\Theta B_1, \end{aligned}$$

Figure 2. Here, it is obvious that from (21). Also, it is also obtained by dividing the vector $\mathcal{F}_1$ by its norm $\left\|\mathcal{F}_1\right\| = \frac{n}{m}\cos(nt)$.

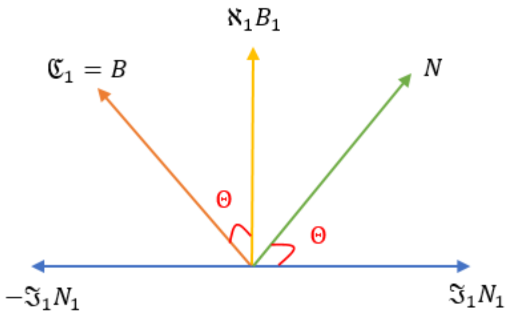

Figure 2. Pole vector $\mathcal{C}_1$ belonging to the type-1 Bishop frame of ψ_m

Corollary 3.4. Binormal vector B and pole vector $\mathcal{C}_1$ belonging to the type-1 Bishop frame of ψ_m are the same.

Proof: The proof is obvious that from (18) and (27).

3.2. Type-2 Bishop Frame of Salkowski Curves in E^3

Theorem 3.6. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, type-2 Bishop frame $\{N_2, B_2, B\}$ of ψ_m obtained by rotating Frenet frame around B by an angle Φ is as follows:

$$\begin{cases} N_2 = \Big(-\sin\Phi\cos t\cos(nt) \\ \qquad -n\sin\Phi\sin t\sin(nt) + \frac{n}{m}\cos\Phi\sin t, \\ \qquad -\sin\Phi\sin t\cos(nt) \\ \qquad +n\sin\Phi\cos t\sin(nt) - \frac{n}{m}\cos\Phi\cos t, \\ \qquad -\frac{n}{m}\sin\Phi\sin(nt) - n\cos\Phi\Big) \\ B_2 = \Big(\cos\Phi\cos t\cos(nt) \\ \qquad +n\cos\Phi\sin t\sin(nt) + \frac{n}{m}\sin\Phi\sin t, \\ \qquad \cos\Phi\sin t\cos(nt) \\ \qquad -n\cos\Phi\cos t\sin(nt) - \frac{n}{m}\sin\Phi\cos t, \\ \qquad \frac{n}{m}\cos\Phi\sin(nt) - n\sin\Phi\Big), \\ B = \Big(-\cos t\sin(nt) + n\sin t\cos(nt) \\ \qquad -\sin t\sin(nt) - n\cos t\cos(nt), \frac{n}{m}\cos(nt)\Big). \end{cases} \tag{27}$$

Proof: The proof is obvious that from (5) and (18).

Corollary 3.5. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, type-2 Bishop frame of ψ_m is obtained by rotating Frenet frame around B by an angle Φ:

$$\Phi = \frac{1}{m}\sin(nt) + c_2, \quad c_2 \in R. \tag{28}$$

Proof: From (5), (17) and (30),

$$\Phi = \int \left\|\psi_m'\right\| \aleph \, dt = \int \frac{n}{m}\cos(nt)\,dt$$
$$= \frac{1}{m}\sin(nt) + c_2, \quad c_2 \in R$$

is obtained.

Corollary 3.6. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, there is the following matrix relation between of type-2 Bishop frame $\{N_2, B_2, B\}$ and Frenet frame $\{T, N, B\}$ of ψ_m:

$$\begin{bmatrix} N_2 \\ B_2 \\ B \end{bmatrix} = \begin{bmatrix} \sin\left(\frac{1}{m}\sin(nt)+c_2\right) & \cos\left(\frac{1}{m}\sin(nt)+c_2\right) & 0 \\ -\cos\left(\frac{1}{m}\sin(nt)+c_2\right) & \sin\left(\frac{1}{m}\sin(nt)+c_2\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Proof: The proof is obvious that from (5) and (28).

Theorem 3.7. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, curvature $\aleph_2$ and torsion $\Im_2$ of type-2 Bishop frame $\{N_2, B_2, B\}$ of ψ_m obtained by rotating Frenet frame around B by an angle Φ are as follows:

$$\begin{cases} \aleph_2 = \tan(nt)\cos\Phi = \tan(nt)\cos\left(\frac{1}{m}\sin(nt)+c_2\right), \\ \Im_2 = \tan(nt)\sin\Phi = \tan(nt)\sin\left(\frac{1}{m}\sin(nt)+c_2\right). \end{cases} \tag{29}$$

Proof: The proof is obvious that from (6), (19) and (28).

Corollary 3.7. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, the derivative vectors of type-2 Bishop frame $\{N_2, B_2, B\}$ of ψ_m are as follows:

$$\begin{cases} N_2{}' = \left(\frac{n}{m}\cos\Phi\cos t\sin^2(nt) \right. \\ \qquad -\frac{n^2}{m}\cos\Phi\sin t\cos(nt)\sin(nt) \\ \qquad \frac{n}{m}\cos\Phi\sin t\sin^2(nt) \\ \qquad +\frac{n^2}{m}\cos\Phi\cos t\cos(nt)\sin(nt), \\ \qquad \left. -\frac{n^2}{m^2}\cos\Phi\cos(nt)\sin(nt) \right), \\ B_2{}' = \left(\frac{n}{m}\sin\Phi\cos t\sin^2(nt) \right. \\ \qquad -\frac{n^2}{m}\sin\Phi\sin t\cos(nt)\sin(nt), \\ \qquad \frac{n}{m}\sin\Phi\sin t\sin^2(nt) \\ \qquad +\frac{n^2}{m}\sin\Phi\cos t\cos(nt)\sin(nt), \\ \qquad \left. -\frac{n^2}{m^2}\sin\Phi\cos(nt)\sin(nt) \right), \\ B' = \frac{n^2}{m^2}\sin(nt)(\sin t, -\cos t, -m). \end{cases} \tag{30}$$

Proof: From (7), (17), (27) and (29), the vectors

$$N_2{}' = -\|\psi_m{}'\|\aleph_2 B,$$

$$B_2{}' = -\|\psi_m{}'\|\Im_2 B,$$

$$B' = \|\psi_m{}'\|\aleph_2 N_2 + \|\psi_m{}'\|\Im_2 B_2$$

are obtained as in (30). These vectors can be obtained in the same way by taking the derivatives of the vectors in (27).

Theorem 3.8. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, the matrix representation of type-2 Bishop derivative formulas of ψ_m:

$$\begin{bmatrix} N_2{}' \\ B_2{}' \\ B' \end{bmatrix} = \frac{n}{m}\sin(nt) \begin{bmatrix} 0 & 0 & -\cos\Phi \\ 0 & 0 & -\sin\Phi \\ \cos\Phi & \sin\Phi & 0 \end{bmatrix} \begin{bmatrix} N_2 \\ B_2 \\ B \end{bmatrix}.$$

Proof: The proof is obvious that from (7), (17) and (29). Also, it is also obtained by comparing expressions (27) and (30).

Theorem 3.9. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, Darboux vector $\mathcal{F}_2$ belonging to the type-2 Bishop frame of ψ_m is as follows:

$$\mathcal{F}_2 = \left(\frac{n}{m}\cos t\cos(nt)\sin(nt) + \frac{n^2}{m}\sin t\sin^2(nt), \right. \\ \frac{n}{m}\sin t\cos(nt)\sin(nt) - \frac{n^2}{m}\cos t\sin^2(nt), \\ \left. \frac{n^2}{m^2}\sin^2(nt) \right) \tag{31}$$

Proof: If (17), (21) and (23) are substituted in (4), (31) is obtained.

Theorem 3.10. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, pole vector $\mathcal{C}_2$ belonging to the type-2 Bishop frame of ψ_m is as follows:

$$\mathcal{C}_2 = \left(\cos t\cos(nt) + n\sin t\sin(nt), \right. \\ \sin t\cos(nt) - n\cos t\sin(nt), \\ \left. \frac{n}{m}\sin(nt) \right). \tag{32}$$

Proof: From (8) and (29), pole vector belonging to the type-2 Bishop frame of ψ_m is

$$\mathcal{C}_2 = \frac{\mathcal{F}_2}{\|\mathcal{F}_2\|} = -\frac{\Im_2}{\sqrt{\aleph_2^2 + \Im_2^2}} N_2 + \frac{\aleph_2}{\sqrt{\aleph_2^2 + \Im_2^2}} B_2$$

$$= -\frac{\Im_2}{\tan(nt)} N_2 + \frac{\aleph_2}{\tan(nt)} B_2$$

$$= \sin\Phi N_2 - \cos\Phi B_2,$$

Figure 3. Here, from (27), it is done. Also, it is also obtained by dividing the vector $\mathcal{F}_2$ by its norm $\|\mathcal{F}_2\| = \frac{n}{m}\sin(nt)$.

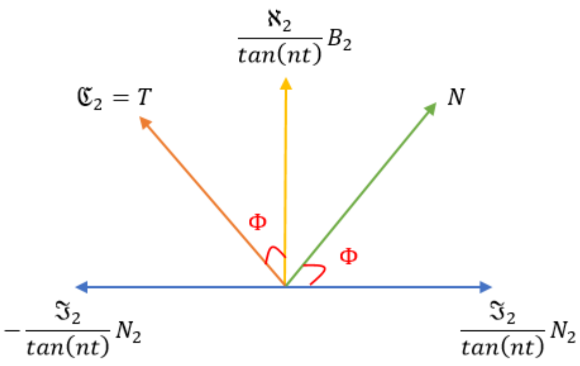

Figure 3. Pole vector $\mathcal{C}_2$ belonging to the type-2 Bishop frame of ψ_m

Corollary 3.8. Tangent vector T and pole vector $\mathcal{C}_2$ belonging to the type-2 Bishop frame of ψ_m are the same.

Proof: From (18) and (34), it is clear.

3.3. Alternative Frame of Salkowski Curves in E^3

Theorem 3.11. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, alternative frame $\{N, C, W\}$ of ψ_m is as follows:

$$\begin{cases} N = \left(\frac{n}{m}\sin t, -\frac{n}{m}\cos t, -n\right), \\ C = (\cos t, \sin t, 0), \\ W = \left(n\sin t, -n\cos t, \frac{n}{m}\right). \end{cases} \tag{33}$$

Proof: The proof is obvious that from (9), (18) and (19).

Corollary 3.9. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, there is the following matrix relation between of alternative frame $\{N, C, W\}$ and Frenet frame $\{T, N, B\}$ of ψ_m:

$$\begin{bmatrix} N \\ C \\ W \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cos(nt) & 0 & -\sin(nt) \\ -\sin(nt) & 0 & \cos(nt) \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{34}$$

Proof: The proof is obvious that from (9) and (19).

Theorem 3.12. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, curvature and torsion of alternative frame $\{N, C, W\}$ of ψ_m are as follows:

$$\begin{cases} F = \frac{1}{\cos(nt)}, \\ G = -n. \end{cases} \tag{35}$$

Proof: The proof is obvious that from (10) and (31). Here, from Definition 2.1, it is seen that $\cos(nt) > 0$.

Corollary 3.10. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, the derivative vectors of alternative frame $\{N, C, W\}$ of ψ_m are as follows:

$$\begin{cases} N' = \left(\frac{n}{m}\cos t, \frac{n}{m}\sin t, 0\right), \\ C' = (-\sin t, \cos t, 0), \\ W' = (n\cos t, n\sin t, 0). \end{cases} \tag{36}$$

Proof: From (11), (17), (33) and (35), the vectors

$$N' = \|\psi_m'\| FC,$$
$$W' = -GC,$$
$$C' = GW - \|\psi_m'\| FN$$

are obtained as in (36). These vectors can be obtained in the same way by taking the derivatives of the vectors in (33).

Theorem 3.13. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, the matrix representation of alternative derivative formulas of ψ_m is as follows:

$$\begin{bmatrix} N' \\ C' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & \frac{n}{m} & 0 \\ -\frac{n}{m} & 0 & n \\ 0 & -n & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix}.$$

Proof: From (11), (17) and (35), it is obtained. Also, it is also obtained by comparing expressions (33) and (36).

Theorem 3.14. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, Darboux vector $\overline{\mathcal{F}}$ belonging to the alternative frame of ψ_m is as follows:

$$\overline{\mathcal{F}} = (0,0,1). \tag{37}$$

Proof: If (17), (33) and (35) are substituted in (12), (37) is obtained.

Theorem 3.15. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, pole vector $\overline{\mathcal{C}}$ belonging to the alternative frame of ψ_m is as follows:

$$\overline{\mathcal{C}} = (0,0,1).$$

Proof: From (12), (17) and (35), pole vector belonging to the alternative frame of ψ_m is

$$\begin{aligned}\overline{\mathcal{C}} = \frac{\overline{\mathcal{F}}}{\left\|\overline{\mathcal{F}}\right\|} &= \frac{G}{\sqrt{G^2+\left\|\psi_m'\right\|^2 F^2}} N + \frac{\left\|\psi_m'\right\| F}{\sqrt{G^2+\left\|\psi_m'\right\|^2 F^2}} W \\ &= GN + \left\|\psi_m'\right\| FW \\ &= -nN + \frac{n}{m} W,\end{aligned}$$

Figure 4. Here, it is obvious that from (33). Also, it is also obtained by dividing the vector $\overline{\mathcal{F}}$ by its norm $\left\|\overline{\mathcal{F}}\right\| = 1$.

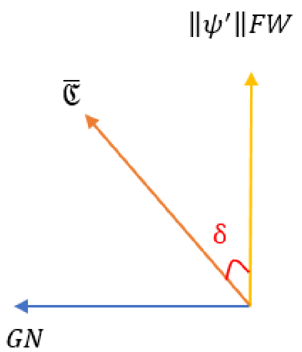

Figure 4. Pole vector $\overline{\mathcal{C}}$ belonging to the alternative frame of ψ_m

Corollary 3.11. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, the angle between of the vectors W and $\overline{\mathcal{C}}$ belonging to the alternative frame of ψ_m is $\delta = arc\tan(m)$.

Proof: From Figure 4, (17) and (35), $\tan\delta = m$. So, the proof is completed.

3.4. N-Bishop Frame of Salkowski Curves in E^3

Theorem 3.16. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, N-Bishop frame $\{N, N_3, B_3\}$ of ψ_m obtained by rotating alternative frame around N by an angle Ω is as follows:

$$\begin{cases} N = \left(\frac{n}{m}\sin t, -\frac{n}{m}\cos t, -n\right), \\ N_3 = \big(\cos\Omega\cos t - n\sin\Omega\sin t, \\ \qquad \cos\Omega\sin t + n\sin\Omega\cos t, -\frac{n}{m}\sin\Omega\big), \\ B_3 = \big(\cos\Omega\cos t - n\sin\Omega\sin t, \\ \qquad \cos\Omega\sin t + n\sin\Omega\cos t, -\frac{n}{m}\sin\Omega\big). \end{cases} \tag{38}$$

Proof: The proof is obvious that from (13) and (18).

Corollary 3.12. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, N-Bishop frame of ψ_m is obtained by rotating alternative frame around N by an angle Ω:

$$\Omega = -nt + c_3, \quad c_3 \in R. \tag{39}$$

Proof: From (13) and (35),

$$\Omega = \int G\, dt = -\int n dt = -nt + c_3, \quad c_3 \in R$$

is obtained.

Corollary 3.13. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, there is the following matrix relation between of N-Bishop frame $\{N, N_3, B_3\}$ and alternative frame $\{N, C, B\}$ of ψ_m:

$$\begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-nt+c_3) & -\sin(-nt+c_3) \\ 0 & \sin(-nt+c_3) & \cos(-nt+c_3) \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix}. \tag{40}$$

Proof: The proof is obvious that from (13) and (38).

Theorem 3.17. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, curvature $\aleph_3$ and torsion $\Im_3$ of N-Bishop frame $\{N, N_3, B_3\}$ of ψ_m obtained by rotating alternative frame around N by an angle Ω are as follows:

$$\begin{cases} \aleph_3 = \dfrac{\cos\Omega}{\cos(nt)} = \dfrac{\cos(-nt+c_3)}{\cos(nt)}, \\ \Im_3 = \dfrac{\sin\Omega}{\cos(nt)} = \dfrac{\sin(-nt+c_3)}{\cos(nt)}. \end{cases} \tag{41}$$

Proof: The proof is obvious that from (14) and (35).

Corollary 3.14. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, the derivative vectors of N-Bishop frame $\{N, N_3, B_3\}$ of ψ_m are as follows:

$$\begin{cases} N' = \left(\dfrac{n}{m}\cos t, \dfrac{n}{m}\sin t, 0\right), \\ N_3' = \dfrac{n^2}{m^2}\cos\Omega(-\sin t, \cos t, -m), \\ B_3' = \dfrac{n^2}{m^2}\sin\Omega(-\sin t, \cos t, m). \end{cases} \tag{42}$$

Proof: From (15), (17), (38) and (41), the vectors

$$N_3' = -\|\psi_m'\|\aleph_3 N,$$

$$B_3' = -\|\psi_m'\|\Im_3 N,$$

$$N' = \|\psi_m'\|\aleph_3 N_3 + \|\psi_m'\|\Im_3 B_3$$

are obtained as in (42). These vectors can be obtained in the same way by taking the derivatives of the vectors in (38).

Theorem 3.18. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, the matrix representation of N-Bishop derivative formulas of ψ_m:

$$\begin{bmatrix} N' \\ N_3' \\ B_3' \end{bmatrix} = \frac{n}{m}\begin{bmatrix} 0 & \cos\Omega & \sin\Omega \\ -\cos\Omega & 0 & 0 \\ -\sin\Omega & 0 & 0 \end{bmatrix} \begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix}.$$

Proof: The proof is obvious that from (15), (17) and (41). Also, it is also obtained by comparing expressions (38) and (41).

Theorem 3.19. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, Darboux vector $\mathcal{F}_3$ belonging to the N-Bishop frame of ψ_m is as follows:

$$\mathcal{F}_3 = \left(\frac{n^2}{m}\sin t, -\frac{n^2}{m}\cos t, \frac{n^2}{m^2}\right). \tag{43}$$

Proof: If (17), (38) and (41) are substituted in (16), (43) is obtained.

Theorem 3.20. For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$, pole vector $\mathcal{C}_3$ belonging to the N-Bishop frame of ψ_m is as follows:

$$\mathcal{C}_3 = \left(n\sin t, -n\cos t, \frac{n}{m}\right). \tag{44}$$

Proof: From (16) and (41), pole vector belonging to the N-Bishop frame of ψ_m is

$$\begin{aligned} \mathcal{C}_3 = \frac{\mathcal{F}_3}{\|\mathcal{F}_3\|} &= -\frac{\Im_3}{\sqrt{\aleph_3^2+\Im_3^2}} N_3 + \frac{\aleph_3}{\sqrt{\aleph_3^2+\Im_3^2}} B_3 \\ &= -\cos(nt)\Im_3 N_3 + \sin(nt)\aleph_3 B_3 \\ &= -\sin\Omega N_3 + \cos\Omega B_3 \end{aligned}$$

Figure 5. Here, from (38), it is done.

Also, it is also obtained by dividing the vector $\mathcal{F}_3$ by its norm $\|\mathcal{F}_3\| = \dfrac{n}{m}$.

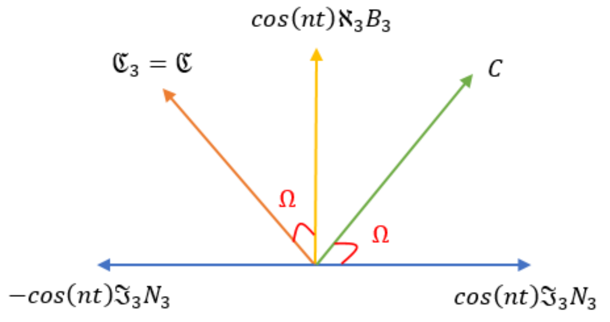

Figure 5. Pole vector $\mathcal{C}_3$ belonging to the type-2 Bishop frame of ψ_m

Corollary 3.15. Pole vector $\mathcal{C}$ and pole vector $\mathcal{C}_3$ belonging to the type-2 Bishop frame of ψ_m are the same.

Proof: From (20) and (44), it is clear.

Corollary 3.16. For $m \neq \pm\dfrac{\sqrt{3}}{3}, 0 \in R$ and $n = \dfrac{m}{\sqrt{m^2+1}}$, there is the following matrix relation between of N-Bishop frame $\{N, N_3, B_3\}$ and Frenet frame $\{T, N, B\}$ of ψ_m:

$$\begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -A(t) & 0 & B(t) \\ B(t) & 0 & A(t) \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

here,

$$A(t) = \cos\Omega\cos(nt) - \sin\Omega\sin(nt),$$
$$B(t) = -\sin\Omega\cos(nt) - \cos\Omega\sin(nt).$$

Proof: The proof is obvious that from (18), (34) and (40).

4. Conclusion and Suggestions

i. The relationship between type-1 Bishop frame $\{T, N_1, B_1\}$ and type-2 Bishop frame $\{N_2, B_2, B\}$ of ψ_m is as follows:

$$\begin{cases} N_2 = \sin\Phi T + \cos\Phi\cos\Theta N_1 + \cos\Phi\sin\Theta B_1, \\ B_2 = -\cos\Phi T + \sin\Phi\cos\Theta N_1 + \sin\Phi\sin\Theta B_1, \\ B = -\sin\Theta N_1 + \cos\Theta B_1. \end{cases}$$

ii. The relationship between type-1 Bishop frame $\{T, N_1, B_1\}$ and N-Bishop frame $\{N, N_3, B_3\}$ of ψ_m is as follows:

$$\begin{cases} N = \cos\Theta N_1 + \sin\Theta B_1, \\ N_3 = \left(-\cos\Omega\cos(nt) + \sin\Omega\sin(nt)\right)T \\ \qquad + \left(\sin\Omega\sin\Theta\cos(nt) + \cos\Omega\sin\Theta\sin(nt)\right)N_1 \\ \qquad - \left(\sin\Omega\cos\Theta\cos(nt) - \cos\Omega\cos\Theta\sin(nt)\right)B_1, \\ B_3 = -\left(\sin\Omega\cos(nt) + \cos\Omega\sin(nt)\right)T \\ \qquad - \left(\cos\Omega\sin\Theta\cos(nt) + \sin\Omega\sin\Theta\sin(nt)\right)N_1 \\ \qquad - \left(\sin\Omega\cos\Theta\cos(nt) - \cos\Omega\cos\Theta\sin(nt)\right)B_1. \end{cases}$$

iii. The relationship between type-1 Bishop frame $\{N_2, B_2, B\}$ and N-Bishop frame $\{N, N_3, B_3\}$ of ψ_m is as follows:

$$\begin{cases} N = \cos\Phi N_2 + \sin\Phi B_2, \\ N_3 = \left(-\cos\Omega\cos(nt) + \sin\Omega\sin(nt)\right)N_2 \\ \qquad + \left(\cos\Omega\cos(nt) - \sin\Omega\sin(nt)\right)B_2 \\ \qquad - \left(\sin\Omega\cos(nt) + \cos\Omega\sin(nt)\right)B, \\ B_3 = -\left(\sin\Omega\cos(nt) + \cos\Omega\sin(nt)\right)N_2 \\ \qquad + \left(\sin\Omega\cos(nt) + \cos\Omega\sin(nt)\right)B_2 \\ \qquad + \left(\cos\Omega\cos(nt) - \sin\Omega\sin(nt)\right)B. \end{cases}$$

In this study, alternative, type-1 Bishop, type-2 Bishop and N-Bishop frames of Salkowski curves in Euclidean 3-space are defined and the theorems and corollaries throughout the paper are obtained through these frames. Thus, it is possible to carry out new studies on these current frames related to the Frenet frame of Salkowski curves. Moreover, similar studies for anti-Salkowski curves or Salkowski curves in Minkowski 3-space are still an open problem.

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