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Kantorovich Stancu Type Operator Including Generalized Brenke Polynomials

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Abstract

This article is concerned with the sequence of operators of Stancu's-type, involving extended Brenke polynomials. We apply Korovkin's theorem to the sequence of positive linear operators, discuss the uniform approximation of continuous functions on closed bounded intervals by known tools theory, and also consider the second modulus of continuity, Peetre's K-functional and Lipschitz class, which are essential concepts in approximation theory.

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1. INTRODUCTION

The idea of approximating functions with polynomials is not only a fundamental aspect of mathematical analysis, but gives valuable mathematical instrument for a wide range of practical fields. The Weierstrass theorem asserts that any function f, is able to expressed using a polynomial series that is uniformly convergent on $[a, b]$. Bernstein provided definitive proof of the Weierstrass approximation theory using a probabilistic construction.

The following operators S_n are introduced and investigated by Szász and known as Szász Mirakjan operators. Agrawal et al. (2018) and Aktaş et al. (2013).

$$S_n(f;x) \coloneqq e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{1.1}$$

Where $n \in N,\ x \geq 0$ and $f \in C[0, \infty)$ whenever the sum mentioned above convergence. These operators are generalizations of Bernstein polynomials to the infinite interval. Several authors have investigated and discussed some approximation properties of the S_n operators.

Varma and Sucu (2022) have given the operators, as follows, (Sucu & Varma, 2019).

$$W_n^*(f;x) = \frac{n}{A(1)\Upsilon(n^2x^2B(1) + nxC(1) + D(1))} \sum_{k=0}^{\infty} \phi_i(nx) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(\varphi)d\varphi,$$

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where

$$A(\vartheta)=\sum_{i=0}^{\infty}a_it^i\ ,\quad B(\vartheta)=\sum_{i=0}^{\infty}b_it^{i+1}\ ,\quad C(\vartheta)=\sum_{i=0}^{\infty}c_it^{i+1}\ ,\quad D(\vartheta)=\sum_{i=0}^{\infty}d_it^{i+3}\ ,\quad \Upsilon(\vartheta)=\sum_{i=0}^{\infty}\tau_it^i$$

are analytic functions on the disc $|z|<\chi$, $\chi>1$. And $a_0\neq 0$, $c_0\neq 0$, $\tau_0\neq 0$. The polynomials ϕ_i are generated by following relation

$$A(1)\Upsilon\big(x^2B(1)+xC(1)+D(1)\big)=\sum_{k=0}^{\infty}\phi_i(x)t^i.$$

Under the assumption $p_k(x)\geq 0$ for $x\epsilon[0,\infty)$, the linear positive operators $P_n(f;x)$ are introduced by Jakimovski and Leviatan (1969), İçöz et al. (2016), Çekim et al. (2019) and Varma (2013).

$$P_n(f;x)\coloneqq\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right), \tag{1.2}$$

where $p_k(x)$ is Appell polynomials and for $n\epsilon N$ and gave the approximation properties of these operators with the help of Szász's method.

The generating functions of the Brenke-type polynomials are as follow Chaggaraa and Gahami (2023) and Atakut and Büyükyazcı (2016).

$$A(t)B(t)=\sum_{k=0}^{\infty}p_k\,(x)t^k,$$

where $p_k(x)$ is Brenke polynomials, A and B are analytic functions by

$$A(t)=\sum_{k=0}^{\infty}a_kt^k\ ,\qquad a_0\neq 0\,,$$

$$B(t)=\sum_{k=0}^{\infty}b_k\,t^k\ ,\quad b_0\neq 0\ (k\geq 0)\,.$$

The following linear positive operators are introduced by Varma et al. (2012) including the Brenke type polynomials (Cheney & Sharma, 1964).

$$\mathcal{L}_n(f;x)\coloneqq\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k\,(nx)f\left(\frac{k}{n}\right), \tag{1.3}$$

where $x\geq 0$ and $n\epsilon\mathbb{N}$. Taking into consideration that

$$g(z)=\sum_{j=0}^{\infty}g_jz^j\qquad ,(g_0\neq 0)$$

the function g is analytic on the set $\{z:|z|<Q\ ,\quad Q>1\}$, and also $g(1)\neq 0$. If polynomials π_k fulfill the subsequent relation

$$g(z)e^{ux}=\sum_{k=0}^{\infty}\pi_k(x)t^k,$$

then it is called Appell polynomials (İçöz & Çekim, 2015). The previously mentioned polynomials hold significant importance as special functions, finding diverse applications in engineering and mathematical analysis.

If $\pi_k(x)$ satisfies the following relation

$$\partial_1(g(t))\partial_2(xg(t)) = \sum_{k=0}^{\infty} \pi_k(x)t^k, \tag{1.4}$$

then it is called generalized Brenke polynomials. İçöz & Çekim (2016a) and Rao et al. (2021), where ∂_1 , ∂_2 and h are functions that are analytic within the specified set

$$\{t: |t| < G\,, \quad G > 1\},$$

such that

$$\begin{aligned} \partial_1(t) &= \sum_{k=0}^{\infty} r_{1,k}t^k\,, \qquad (r_{1,0} \neq 0) \\ \partial_2(t) &= \sum_{k=0}^{\infty} r_{2,k}t^k\,, \qquad (r_{2,0} \neq 0) \\ h(t) &= \sum_{k=0}^{\infty} h(t)t^k\,, \qquad (h(1) \neq 0) \end{aligned} \tag{1.5}$$

under the following assumptions:

(i) $\partial_1(1) \neq 0, \dfrac{a_{k-\tau}b_\tau}{\partial(1)} \geq 0, \quad 0 \leq \tau \leq k, \quad k = 0,1,2,\ldots$

(ii) $B: [0,\infty) \longrightarrow (0,\infty)$

(iii) (1.6) and (1.7) converge for $|t| < G, (G > 1)$ where

$$\partial(t) = \sum_{\tau=0}^{\infty} r_\tau t^\tau\,, \qquad r_0 \neq 0 \tag{1.6}$$

$$B(t) = \sum_{\tau=0}^{\infty} b_\tau t^\tau\,, \qquad b_\tau \neq 0\ (\tau \geq 0) \tag{1.7}$$

are analytic functions, and the Kantorovich type of the operators. İçöz & Çekim (2016b), under the above assumptions, is defined as

$$K_n(f;x) \coloneqq \frac{n}{\partial(1)B(nx)} \sum_{k=0}^{\infty} p_k\,(nx) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt$$

where $n \in \mathbb{N}\,, x \geq 0\,, f \in C[0,\infty)$, and $p_k(x)$ is the Brenke type polynomials.

Sucu (2022) has given sequence of operators $\{L_n^{(\sigma_1,\sigma_2)}\}_{n\geq 1}\,, \sigma_1, \sigma_2 \geq 0$ as follows:

$$L_n^{(\sigma_1,\sigma_2)}(f;x) \coloneqq \frac{1}{\partial_1(g(1))\partial_2(nxg(1))} \sum_{k=0}^{\infty} \pi_k\,(nx) f\left(\frac{k+\sigma_1}{n+\sigma_2}\right), \tag{1.8}$$

$\pi_k(x)$ polynomials are given by (1.4), and also the polynomials $\pi_k(x)$ and function h satisfies these conditions $h'(1) = 1\,, \pi_k(x) \geq 0$.

Motivated by the above-mentioned works, we consider the Kantorovich Stancu type linear positive operators as follows:

$$L_m^{*(\sigma_1,\sigma_2)}(f;x) \coloneqq \frac{m+\sigma_2}{\partial_1(g(1))\partial_2(mxg(1))} \sum_{k=0}^{\infty} \pi_k\,(mx) \int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}} f(\varphi)d\varphi. \tag{1.9}$$

Where $f \epsilon C[0,\infty)$ and $x \epsilon [0,\infty)$.

Here, we assume that

(i) $\partial_1, \partial_2: \mathbb{R} \longrightarrow (0,\infty)$

(ii) (1.4) and (1.5) converge for $|t| < G, (G > 1)$.

This article focuses on the operators of Stancu-type, containing extended Brenke polynomials. the secound part provides, some results which are essential for $L_m^{*(\sigma_1,\sigma_2)}$ and in the third section, we employ Korovkin theorem, we are discussing and considering the second modulus of continuity, Peetre's K- functional in the continuous functions space and Lipschitz class, which are fundamental concepts in approximation theory.

2. SOME RESULTS OF THE OPERATORS $L_m^{*(\sigma_1,\sigma_2)}$

Before that, to apply the Korovkin-type theorem to the sequence of operators $\{L_m^{*(\sigma_1,\sigma_2)}\}_{m\geq 1}$ we need to have some essential lemmas.

Lemma 2.1. If π_k is polynomials fulfilling the equality (1.4), then

$$(i)\quad \sum_{k=0}^{\infty}\pi_k(mx)=\partial_1(g(1))\partial_2(mxg(1))$$

$$(ii)\quad \sum_{k=0}^{\infty}k\pi_k(mx)=\partial'_1(g(1))\partial_2(mxg(1))+mx\partial_1(g(1))\partial'_2(mxg(1))$$

$$(iii)\quad \sum_{k=0}^{\infty}k^2\pi_k(mx)=\left[(g''(x)+1)\partial'_1(g(1))+\partial''_1(g(1))\right]\partial_2(mxg(1))$$
$$+\left[2\partial'_1(g(1))+(g''(1)+1)\partial_1(g(1))\right]\partial'_2(mxg(1))mx$$
$$+\partial_1(g(1))\partial''_2(mxg(1))(mx)^2$$

Proof. Using the relation given by(1.4), we have

$$\partial_1(g(t))\partial_2(mxg(t))=\sum_{k=0}^{\infty}\pi_k(mx)t^k.$$

For $t=1$ we have

$$\partial_1(g(1))\partial_2(xmg(1))=\sum_{k=0}^{\infty}\pi_k(mx).$$

From the relation $\sum_{k=0}^{\infty}\pi_k(mx)t^k=\partial_1(g(t))\partial_2(mxg(t))$, we take the derivative from the both side of equality and get

$$\sum_{k=0}^{\infty}k\pi_k(mx)t^{k-1}=g'(t)\partial'_1(g(t))\partial_2(mxg(t))+mxg(t)\partial_1(g(t))\partial'_2(mxg(t)).$$

If $t=1$and $g'(1)=1$, then we obtain

$$\sum_{k=0}^{\infty}k\pi_k(mx)=\partial_2(mxg(1))\partial'_1(g(1))+mx\partial_1(g(1))\partial'_2(mxg(1)).$$

As we know

$$\sum_{k=0}^{\infty}k\pi_k(mx)t^{k-1}=g'(t)\partial'_1(g(t))\partial_2(mxg(t))+mxg'(t)\partial_1(g(t))\partial'_2(mxg(t)),$$

then we have

$$\sum_{k=0}^{\infty} k(k-1)\pi_k(mx)t^{k-2} = \partial''_1\big(g(t)\big)g'(t)\partial_2\big(mxg(t)\big) + \partial'_1\big(g(t)\big)g''(t)\partial_2\big(mxg(t)\big)$$

$$+\partial'_1\big(g(t)\big)g'(t)mxg'(t)\partial'_2\big(mxg(t)\big) + mxg''(t)\partial_1\big(g(t)\big)\partial'_2\big(mxg(t)\big)$$

$$+mxg'(t)g'(t)\partial'_1\big(g(t)\big)\partial'_2\big(mxg(t)\big)$$

$$+(mx)^2g'(t)\partial_1\big(g(t)\big)g'(t)\partial''_2\big(mxg(t)\big).$$

If $t = 1$and $g'(1) = t$, then we acquire

$$\sum_{k=0}^{\infty} k^2\pi_k(mx) - \sum_{k=0}^{\infty} k\pi_k(mx) = \partial''_1\big(g(1)\big)\partial_2\big(mxg(1)\big) + \partial'_1\big(g(1)\big)g''(1)\partial_2\big(mxg(1)\big)$$

$$+mx\partial'_1\big(g(1)\big)\partial'_2\big(mxg(1)\big) + mxg''(1)\partial_1\big(g(1)\big)\partial'_2\big(mxg(1)\big)$$

$$+mx\partial'_1\big(g(t)\big)\partial'_2\big(mxg(1)\big) + (mx)^2\partial_1\big(g(1)\big)\partial''_2\big(mxg(1)\big).$$

So, we win

$$\sum_{k=0}^{\infty} k^2\pi_k(mx) = \partial''_1\big(g(1)\big)\partial_2\big(mxg(1)\big) + \partial'_1\big(g(1)\big)h''(1)\partial_2\big(mxg(1)\big)$$

$$+mx\partial'_1\big(g(1)\big)\partial'_2\big(mxg(1)\big) + mxg''(1)\partial_1\big(g(1)\big)\partial'_2\big(mxg(1)\big)$$

$$+mx\partial'_1\big(g(t)\big)\partial'_2\big(mxg(1)\big) + (mx)^2\partial_1\big(g(1)\big)\partial''_2\big(mxg(1)\big) + \sum_{k=0}^{\infty} k\pi_k(mx)$$

$$\sum_{k=0}^{\infty} k^2\pi_k(mx) = \partial_2\big(mxg(1)\big)\partial''_1\big(g(1)\big) + g''(1)\partial'_1\big(g(1)\big)\partial_2\big(mxg(1)\big)$$

$$+mx\partial'_1\big(g(1)\big)\partial'_2\big(mxg(1)\big) + mxg''(1)\partial_1\big(g(1)\big)\partial'_2\big(mxg(1)\big)$$

$$+mx\partial'_1\big(g(t)\big)\partial'_2\big(mxg(1)\big) + (mx)^2\partial_1\big(g(1)\big)\partial''_2\big(mxg(1)\big)$$

$$+\partial'_1\big(g(1)\big)\partial_2\big(mxg(1)\big) + mx\partial_1\big(g(1)\big)\partial'_2\big(mxg(1)\big)$$

$$= \big[\partial'_1\big(g(1)\big)(g''(1)+1) + \partial''_1\big(g(1)\big)\big]\partial_2\big(mxg(1)\big)$$

$$+\big[2\partial'_1\big(g(1)\big) + \partial_1\big(g(1)\big)(g''(1)+1)\big]\partial'_2\big(mxg(1)\big)mx$$

$$+\partial_1\big(g(1)\big)\partial''_2\big(mxg(1)\big)(mx)^2.$$

Lemma 2.2. For $m \geq 1$ we notch-up the following identities:

(i) $\quad L_m^{*(\sigma_1,\sigma_2)}(1;x) = 1,$

(ii) $\quad L_m^{*(\sigma_1,\sigma_2)}(s;x) = \frac{1}{m+\sigma_2}\left(\frac{m\partial'_2(mxg(1))}{\partial_2(mxg(1))}x + \frac{\partial'_1(g(1))}{\partial_1(g(1))} + \frac{2v_1+1}{2}\right),$

(iii) $L_m^{*(\sigma_1,\sigma_2)}(s^2;x) = \frac{1}{(m+\sigma_2)^2}\left\{\frac{m^2\partial''_2\big(mxg(1)\big)}{\partial_2\big(mxg(1)\big)}x^2\right.$

$$+\left(\frac{m\mathcal{g}'_2\big(mx\mathcal{g}(1)\big)\big[\partial'_1\big(\mathcal{g}(1)\big)+\big(\mathcal{g}''(1)+1\big)\partial_1\big(d(1)\big)\big]}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)}\right)x$$

$$+\frac{(\mathcal{g}''(1)+1)\partial'_1\big(\mathcal{g}(1)\big)+\partial''_1\big(\mathcal{g}(1)\big)+(2\sigma_1+1)\partial'_1\big(\mathcal{g}(1)\big)}{\partial_1\big(\mathcal{g}(1)\big)}+\frac{3\sigma_1^2+3\sigma_1+1}{3}\Bigg\}.$$

Proof. Using the operator given by (1.9), we have

$$L_m^{*(\sigma_1,\sigma_2)}(1;x)=\frac{m+\sigma_2}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)}\sum_{k=0}^{\infty}\pi_k(mx)\int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+v_2}}d\varphi$$

$$=\frac{m+\sigma_2}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)}\sum_{k=0}^{\infty}\pi_k(mx)\left(\varphi\Big|_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+v_1+1}{m+\sigma_2}}\right)$$

$$=\frac{m+\sigma_2}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)}\sum_{k=0}^{\infty}\pi_k(mx)\frac{1}{m+\sigma_2}.$$

As we know from Lemma 2.1.

$$\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)=\sum_{k=0}^{\infty}\pi_k(mx),$$

then we attain

$$L_m^{*(v_1,v_2)}(1;x)=1.$$

Using the operator (1.9), we get.

$$L_m^{*(\sigma_1,\sigma_2)}(s;x)=\frac{m+\sigma_2}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)}\sum_{k=0}^{\infty}\pi_k(mx)\int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}}\varphi\,d\varphi$$

$$=\frac{m+\sigma_2}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)}\sum_{k=0}^{\infty}\pi_k(mx)\frac{1}{2}\left(\varphi^2\Big|_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+v_1+1}{m+v_2}}\right)$$

$$=\frac{m+\sigma_2}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)}\sum_{k=0}^{\infty}\pi_k(mx)\left\{\frac{k}{(m+\sigma_2)^2}+\frac{2\sigma_1+1}{2(m+\sigma_2)^2}\right\}$$

$$=\frac{m+\sigma_2}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)}\left\{\sum_{k=0}^{\infty}\pi_k(mx)\frac{k}{(m+\sigma_2)^2}+\sum_{k=0}^{\infty}\pi_k(mx)\frac{2\sigma_1+1}{2(m+\sigma_2)^2}\right\}$$

$$=\frac{m+\sigma_2}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)}\left\{\frac{1}{(m+\sigma_2)^2}\sum_{k=0}^{\infty}k\pi_k(mx)+\frac{2v_1+1}{2(m+v_2)^2}\sum_{k=0}^{\infty}\pi_k(mx)\right\}$$

$$=\frac{1}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx(1)\big)}\left\{\frac{1}{m+\sigma_2}\Big(\partial'_1\big(h(1)\big)\partial_2\big(mxh(1)\big)+mx\partial'_1\big(\mathcal{g}(1)\big)\,\partial'_2\big(mx\mathcal{g}(1)\big)\Big)\right.$$
$$\left.+\frac{2\sigma_1+1}{2(m+\sigma_2)}\partial_1\big(\mathcal{g}(1)\big)\partial_2(mxh\mathcal{g})\right\}$$

$$=\frac{\partial'_1(g(1))}{\partial_1(g(1))(m+\sigma_2)}+\frac{m\partial'_2(mxg(1))}{\partial_2(mxg(1))(m+\sigma_2)}x+\frac{2\sigma_1+1}{2(m+\sigma_2)}$$
$$=\frac{1}{m+\sigma_2}\left(\frac{\partial'_1(g(1))}{\partial_1(g(1))}+\frac{m\partial'_2(mxg(1))}{\partial_2(mxg(1))}x+\frac{2\sigma_1+1}{2}\right).$$

From (1.9), we come by

$$L_m^{*(\sigma_1,\sigma_2)}(s^2;x)=\frac{m+\sigma_2}{\partial_1(g(1))\partial_2(mxg(1))}\sum_{k=0}^{\infty}\pi_k(mx)\int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}}\varphi^2\,d\varphi$$
$$=\frac{m+\sigma_2}{\partial_1(g(1))\partial_2(mxg(1))}\sum_{k=0}^{\infty}\pi_k(mx)\frac{1}{3}\left(\varphi^3\Big|_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}}\right)$$
$$=\frac{m+\sigma_2}{\partial_1(g(1))\partial_2(mxg(1))}\sum_{k=0}^{\infty}\pi_k(mx)\left\{\frac{k^2}{(m+\sigma_2)^3}+\frac{k(2\sigma_1+1)}{(m+\sigma_2)^3}+\frac{3\sigma_1^2+3\sigma_1+1}{3(m+\sigma_2)^3}\right\}$$
$$=\frac{1}{\partial_1(h(1))\partial_2(mxh(1))}\left\{\frac{1}{(m+\sigma_2)^2}\sum_{k=0}^{\infty}k^2\pi_k(mx)+\frac{2\sigma_1+1}{(m+\sigma_2)^2}\sum_{k=0}^{\infty}k\pi_k(mx)\right.$$
$$\left.+\frac{3\sigma_1^2+3\sigma_1+1}{3(m+\sigma_2)^2}\sum_{k=0}^{\infty}\pi_k(mx)\right\}$$

If we put the values from the Lemma 2.1 in the last equality, then we get

$$L_m^{*(\sigma_1,\sigma_2)}(s^2;x)=\frac{1}{(m+\sigma_2)^2}\left\{\frac{m^2\partial''_2(mxg(1))}{\partial_2(mxh(1))}x^2\right.$$
$$+\left(\frac{m\partial''_2(mxg(1))[\partial'_1(g(1))+(g''(1)+1)\partial_1(g(1))]}{\partial_1(g(1))\partial_2(mxg(1))}+\frac{m(2\sigma_1+1)\partial'_2(mxg(1))}{\partial_2(mxg(1))}\right)x$$
$$\left.+\frac{(g''(1)+1)\partial'_1(g(1))+\partial''_1(g(1))+(2\sigma_1+1)\partial'_1(g(1))}{\partial_1(g(1))}+\frac{3\sigma_1^2+3\sigma_1+1}{3}\right\}.$$

Lemma 2.3. Let $L_m^{*(v_1,v_2)}$ be operators which are defined by (1.9), then it follows that:

$$(i)\quad L_m^{*(\sigma_1,\sigma_2)}(s-x;x)=\frac{1}{m+\sigma_2}\left\{\left(\frac{m\partial'_2(mxg(1))}{\partial_2(mxg(1))}-(m+\sigma_2)\right)x+\frac{\partial'_1(g(1))}{\partial_1(g(1))}+\frac{2\sigma_1+1}{2}\right\}$$

$$(ii)\quad L_n^{*(\sigma_1,\sigma_2)}((s-x)^2;x)=\left(\frac{m^2\partial''_2(mxg(1))}{\partial_2(mxg(1))(m+\sigma_2)^2}-\frac{2m\partial'_2(mxg(1))}{\partial_2(mxg(1))(m+\sigma_2)}+1\right)x^2$$
$$+\left(\frac{m[\partial'_1(g(1))+(g''(1)+1)\partial''_2(mxg(1))]}{\partial_1(g(1))\partial_2(mxg(1))(m+\sigma_2)^2}+\frac{m(2\sigma_1+1)\partial'_2(mxg(1))}{(m+\sigma_2)^2\partial_2(mxg(1))}\right.$$
$$\left.-\frac{2\partial'_1(g(1))}{(m+\sigma_2)\partial_1(g(1))}-\frac{2\sigma_1+1}{m+\sigma_2}\right)x$$
$$+\frac{(g''(1)+1)\partial_1(g(1))+(g''(1)+1)\partial''_1(g(1))+(2\sigma_1+1)\partial'_1(g(1))}{(m+\sigma_2)^2\partial_1(g(1))}$$
$$+\frac{3\sigma_1^2+3\sigma_1+1}{3(m+\sigma_2)^2}.$$

Proof. According to the principle of linearity of $L_m^{*(\sigma_1,\sigma_2)}$ operators and applying Lemma 2.2

$$L_m^{*(\sigma_1,\sigma_2)}(s-x;x) = L_m^{*(\sigma_1,\sigma_2)}(s;x) - L_m^{*(\sigma_1,\sigma)}(x;x) = L_m^{*(\sigma_1,\sigma_2)}(s;x) - xL_m^{*(\sigma_1,\sigma_2)}(1;x)$$

$$= \left(\frac{m\partial'_2\big(mx\mathcal{g}(1)\big)}{\partial_2\big(mxh(1)\big)}x + \frac{\partial'_1\big(\mathcal{g}(1)\big)}{\partial_1\big(h(1)\big)} + \frac{2\sigma_1+1}{2}\right)\frac{1}{m+\sigma_2} - x$$

$$= \left(\frac{m\partial'_2\big(mx\mathcal{g}(1)\big)}{\partial_2\big(mx\mathcal{g}(1)\big)(m+\sigma_2)} - 1\right)x + \frac{1}{m+\sigma_2}\left(\frac{\partial'_1\big(\mathcal{g}(1)\big)}{\partial_1\big(\mathcal{g}(1)\big)} + \frac{2\sigma_1+1}{2}\right),$$

and

$$L_m^{*(\sigma_1,\sigma_2)}((s-x)^2;x) = L_m^{*(\sigma_1,\sigma_2)}(s^2-2sx+x^2;\ x\ )$$

$$= L_m^{*(\sigma_1,\sigma_2)}(s^2;x) - 2xL_m^{*(\sigma_1,\sigma_2)}(s;x) + x^2$$

$$= \frac{1}{(m+\sigma_2)^2}\left\{\frac{m^2\partial''_2(mx\mathcal{g}(1))}{\partial_2(mx\mathcal{g}(1))}x^2\right.$$

$$+\left(\frac{m\big[\partial'_1\big(\mathcal{g}(1)\big) + (\mathcal{g}''(1)+1)\partial_1\big(\mathcal{g}(1)\big)\big]\partial''_2\big(mx\mathcal{g}(1)\big)}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)}\right.$$

$$\left.+\frac{m(2\sigma_1+1)\partial'_2\big(mx\mathcal{g}(1)\big)}{\partial_2\big(mxh(1)\big)}\right)x$$

$$+\frac{(\mathcal{g}''(1)+1)\partial'_1\big(\mathcal{g}(1)\big) + \partial''_1\big(\mathcal{g}(1)\big) + (2\sigma_1+1)\partial'_1\big(\mathcal{g}(1)\big)}{\partial_1\big(\mathcal{g}(1)\big)}$$

$$\left.+\frac{3\sigma_1^2+3\sigma_1+1}{3}\right\}$$

$$-\frac{2}{m+\sigma_2}\left(\frac{m\partial'_2\big(mx\mathcal{g}(1)\big)}{\partial_2\big(mx\mathcal{g}(1)\big)}x^2 + \frac{\partial'_1\big(\mathcal{g}(1)\big)}{\partial_1\big(\mathcal{g}(1)\big)}x + \frac{2\sigma_1+1}{2}x\right) + x^2$$

$$= \left(\frac{m^2\partial''_2\big(mx\mathcal{g}(1)\big)}{\partial_2\big(mx\mathcal{g}(1)\big)(m+\sigma_2)^2} - \frac{2m\partial'_2\big(mx\mathcal{g}(1)\big)}{\partial_2\big(mx\mathcal{g}(1)\big)(m+\sigma_2)} + 1\right)x^2$$

$$+\left(\frac{m\big[\partial'_1\big(\mathcal{g}(1)\big) + (\mathcal{g}''(1)+1)\partial''_2\big(mx\mathcal{g}(1)\big)\big]}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)(m+\sigma_2)^2} + \frac{m(2\sigma_1+1)\partial'_2\big(mx\mathcal{g}(1)\big)}{(m+\sigma_2)^2\partial_2\big(mx\mathcal{g}(1)\big)}\right.$$

$$\left.-\frac{2\partial'_1\big(\mathcal{g}(1)\big)}{(m+\sigma_2)\partial_1\big(\mathcal{g}(1)\big)} - \frac{2\sigma_1+1}{m+\sigma_2}\right)x$$

$$+\frac{(\mathcal{g}''(1)+1)\partial_1\big(\mathcal{g}(1)\big) + (\mathcal{g}''(1)+1)\partial''_1\big(\mathcal{g}(1)\big) + (2v_1+1)\partial'_1\big(\mathcal{g}(1)\big)}{(m+\sigma_2)^2\partial_1\big(\mathcal{g}(1)\big)}$$

$$+\frac{3\sigma_1^2+3\sigma_1+1}{3(m+\sigma_2)^2}.$$

Now we define

$$M_{1,m}^{(\sigma_1,\sigma_2)} := L_m^{*(\sigma_1,\sigma_2)}(s-x;x)$$

$$M_{2,m}^{(\sigma_1,\sigma_2)} := L_m^{*(\sigma_1,\sigma_2)}((s-x)^2;x) \tag{2.1}$$

Now we define $\tilde{C}[0,\infty)$ and $C_B[0,\infty)$ to represent the set of all uniformly continuous functions and bounded, continuous functions on $[0,\infty)$ respectively. With this we can now determine the arrange of approximation for the functions within the space $\tilde{C}[0,\infty) \cap E$ where

$$E = \left\{f: \frac{f(x)}{1+x^2} \text{ } is\ convergent\ as\ x \longrightarrow \infty\right\}.$$

Theorem 2.1. Let $f:[0,\infty) \longrightarrow \mathbb{R}$ be a continuous function which belongs to class E and

$$\lim_{y\to\infty} \frac{\partial'_2(y)}{\partial_2(y)} = 1 \quad and \quad \lim_{y\to\infty} \frac{\partial''_2(y)}{\partial_2(y)} = 1. \tag{2.2}$$

Then we see that

$$L_m^{*(\sigma_1,\sigma_2)}(f;x) \longrightarrow f(x)$$

uniformly as $m \longrightarrow \infty$ over every limited subset within the interval $[0,\infty)$.

Proof. Under the assumption of (2.2) and using Lemma 2.2 for $i = 0,1,2$, we acquire the subsequent equality

$$L_m^{*(\sigma_1,\sigma_2)}(s^i;x) \longrightarrow x^i$$

for $i = 0$ which is

$$L_m^{*(\sigma_1,\sigma_2)}(s^i;x) = L_m^{*(\sigma_1,\sigma_2)}(1;x) = 1.$$

So, we have

$$\lim_{m\to\infty} \left\{L_m^{*(\sigma_1,\sigma_2)}(1;x)\right\} = 1.$$

For $i = 1$

$$L_m^{*(\sigma_1,\sigma_2)}(s^i;x) = L_m^{*(\sigma_1,\sigma_2)}(s;x) = \frac{1}{m+\sigma_2}\left(\frac{m\partial'_2(mxg(1))}{\partial_2(mxg(1))}x + \frac{\partial'_1(g(1))}{\partial_1(g(1))} + \frac{2\sigma_1+1}{2}\right),$$

and we come off

$$\lim_{m\to\infty} \left\{L_m^{*(\sigma_1,\sigma_2)}(s;x)\right\} = \lim_{m\to\infty} \left\{\frac{1}{m+\sigma_2}\left(\frac{m\partial'_2(mxg(1))}{\partial_2(mxg(1))}x + \frac{\partial'_1(g(1))}{\partial_1(g(1))} + \frac{2\sigma_1+1}{2}\right)\right\}$$

$$= \lim_{m\to\infty} \frac{\partial'_2(mxg(1))}{\partial_2(mxg(1))}x = x \lim_{y\to\infty} \frac{\partial'_2(y)}{\partial_2(y)} = x.$$

Finally, for $i = 2$

$$\lim_{m\to\infty} \left\{L_m^{*(\sigma_1,\sigma_2)}(s^2;x)\right\} = \lim_{m\to\infty} \frac{1}{(m+\sigma_2)^2}\left\{\frac{m^2\partial''_2(mxg(1))}{\partial_2(mxg(1))}x^2\right.$$

$$+\left(\frac{m\left[\partial'_1(g(1)) + (g''(1)+1)\partial_1(g(1))\right]\partial''_2(mxg(1))}{\partial_1(g(1))\partial_2(mxg(1))}\right.$$

$$\left.+\frac{m(2\sigma_1+1)\partial'_2(mxg(1))}{\sigma_2(mxg(1))}\right)x$$

$$\left.+\frac{(g''(1)+1)\partial'_1(g(1)) + \partial''_1(g(1)) + (2\sigma_1+1)\partial'_1(g(1))}{\partial_1(g(1))} + \frac{3\sigma_1^2+3\sigma_1+1}{3}\right\}$$

$$= \lim_{m\to\infty} \frac{\partial''_2(mx\mathcal{g}(1))}{\partial_2(mx\mathcal{g}(1))} x^2 = x^2 \lim_{m\to\infty} \frac{\partial''_2(y)}{\partial_2(y)} = x^2.$$

In the result, we come off

$$L_m^{*(\sigma_1,\sigma_2)}(s^i;x) \longrightarrow x^i \ , \qquad i = 0,1,2.$$

3. APPROXIMATION TO FUNCTIONS USING $L_m^{*(\sigma_1,\sigma_2)}$ OPERATORS

Theorem 3.1. Let f be a function within the class $f \in \tilde{C}[0,\infty) \cap E$, then

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| \le 2\omega\big(f;\delta_m(x)\big),$$

where $\delta_m(x) = \sqrt{M_{2,m}^{(\sigma_1,\sigma_2)}(x)}$ and $\omega(f;.)$ is modulus of continuity of the function f.

Proof. By using the operator (1.9), we have

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| = \left|\frac{1}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)} \sum_{k=0}^{\infty} \pi_k\,(mx) \int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}} \big(f(\varphi) - f(x)\big) d\varphi\right|$$

$$\le \frac{1}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)} \sum_{k=0}^{\infty} \pi_k\,(mx) \int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma+1}{m+\sigma}} |f(\varphi) - f(x)| d\varphi.$$

By using the property of the modulus of continuity, we have

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| \le \frac{1}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)} \sum_{k=0}^{\infty} \pi_k\,(mx) \int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}} \left(1 + \frac{1}{\delta}|\varphi - x|\right) \omega(f;\delta) d\varphi$$

$$= \frac{\omega(f;\delta)}{\partial_1\big(\mathcal{g}(1)\big)\partial_2\big(mx\mathcal{g}(1)\big)} \left(\sum_{k=0}^{\infty} \pi_k\,(mx) \int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+v_2}} d\varphi\right.$$

$$\left. + \sum_{k=0}^{\infty} \pi_k\,(mx) \frac{1}{\delta} \int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}} |\varphi - x| d\varphi\right)$$

$$= \left(L_m^{*(\sigma_1,\sigma_2)}(1;x) + \frac{1}{\delta} L_m^{*(\sigma_1,\sigma_2)}(|s-x|;x)\right) \omega(f;\delta)$$

$$= \left(1 + \frac{1}{\delta} L_m^{*(\sigma_1,\sigma_2)}(|s-x|;x)\right) \omega(f;\delta)$$

$$= \left(1 + \frac{1}{\delta}\delta\right) \omega(f;\delta) = 2\omega(f;\delta_m(x)).$$

In the result, we get

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| \le 2\omega\big(f;\delta_m(x)\big).$$

Definition 1. If for $0 < \alpha \leq 1$ and $M > 0$ function f satisfy the following inequality, then it is called to be Lipschitz property of arrange α

$$|f(t) - f(x)| \leq M|t - x|^{\alpha}, \quad t, x \in [0, \infty)$$

the set of all function that belongs to Lipschitz class functions is given as follows (Srivasta et al., 2019).

$$Lip_M(\alpha) = \{f : |f(t) - f(x)| \leq M|t - x|^{\alpha} \ , \qquad t, x \in [0, \infty)\}.$$

Theorem 3.2. Let f be a function of set $Lip_M(\alpha)$, then for $x \geq 0$

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| \leq M\delta_m^{\alpha}(x)$$

where $\delta_m(x) = \sqrt{M_{2,m}^{(\sigma_1,\sigma_2)}(x)}$.

Proof. By using the operator (1.9), we have

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| = \left|L_m^{*(\sigma_1,\sigma_2)}(f(s) - f(x);x)\right|$$

$$= \left|\frac{1}{\partial_1(g(1))\partial_2(mxg(1))}\sum_{k=0}^{\infty}\pi_k(mx)\int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}}\big(f(\varphi) - f(x)\big)d\varphi\right|$$

$$\leq \frac{1}{\partial_1(g(1))\partial_2(mxg(1))}\sum_{k=0}^{\infty}\pi_k(mx)\int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}}|f(\varphi) - f(x)|d\varphi.$$

As f is a function of $Lip_M(\alpha)$ then

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| \leq \frac{1}{\partial_1(g(1))\partial_2(mxg(1))}\sum_{k=0}^{\infty}\pi_k(mx)\int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}}M|\varphi - x|^{\alpha}d\varphi.$$

By application of Holder`s inequality we obtain

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| \leq M\left\{\frac{1}{\partial_1(g(1))\partial_2(mxg(1))}\sum_{k=0}^{\infty}(\pi_k(mx))^{\frac{2-\alpha}{2}}(\pi_k(mx))^{\frac{\alpha}{2}}\int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}}|\varphi - x|^{\alpha}d\varphi\right\}$$

$$\leq M\left\{\frac{1}{\partial_1(g(1))\partial_2(mxg(1))}\big[\partial_1(g(1))\partial_2(mxg(1))\big]^{\frac{2-\alpha}{2}}\right.$$

$$\times\left[\frac{1}{\partial_1(g(1))\partial_2(mxg(1))}\sum_{k=0}^{\infty}\pi_k(mx)\right]^{\frac{2-\alpha}{2}}\big[\partial_1(g(1))\partial_2(mxg(1))\big]^{\frac{\alpha}{2}}$$

$$\left.\times\left[\frac{1}{\partial_1(g(1))\partial_2(mxg(1))}\sum_{k=0}^{\infty}\pi_k(mx)\int_{\frac{k+\sigma_1}{m+\sigma_2}}^{\frac{k+\sigma_1+1}{m+\sigma_2}}|\varphi - x|^2d\varphi\right]^{\frac{\alpha}{2}}\right\}$$

$$\leq M\left\{L_m^{*(\sigma_1,\sigma_2)}(1;x)\right\}^{\frac{2-\alpha}{2}}\left\{L_m^{*(\sigma_1,\sigma_2)}((s-x)^2;x)\right\}^{\frac{\alpha}{2}}$$

$$= M\left(\sqrt{M_{2,m}^{(\sigma_1,\sigma_2)}(x)}\right)^{\alpha}.$$

In the result, we get

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| \leq M\delta_m^{\alpha}(x).$$

Definition 2. If $f \in C_B[0,\infty)$, then the Peetre's K- functional for f is defined by

$$K(f;\delta) = inf\left\{||f-g|| + \delta||g||_{C_B^2[0,\infty)}\right\}$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty): g', g'' \in C_B[0,\infty)\}$ with the norm

$$||g||_{C_B^2[0,\infty)} = ||g||_{C_B[0,\infty)} + ||g'||_{C_B[0,\infty)} + ||g''||_{C_B[0,\infty)}.$$

A norm of linear space $C_B[0,\infty)$ is defined by Sucu et al. (2012),

$$||f||_{C_B[0,\infty)} = \sup_{x\in[0,\infty)} |f(x)|.$$

Definition 3. The second modulus of continuity of the function $f \in C_B[0,\infty)$ is defined by

$$\omega_2 := \sup_{0<t\leq\delta} \|f(.+2t) - 2f(.+t) + f(.)\|_{C_B[0,\infty)}.$$

It is known that there exists a constant $C > 0$ such that (İçöz & Eryiğit, 2020).

$$K(f;\delta) \leq C\omega_2\big(f;\sqrt{\delta}\big). \tag{3.1}$$

Theorem 3.3. Suppose $f \in C_B[0,\infty)$ and $x \in [0,\infty)$. Then

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| \leq 2K\big(f;\lambda_m(x)\big)$$

where

$$\lambda_m(x) = \frac{1}{2}\left[M_{1,m}^{(\sigma_1,\sigma_2)}(x) + M_{2,m}^{(\sigma_1,\sigma_2)}(x)\right]$$

and $K(f;.)$ represents the Peetre's K -functional for the function f.

Proof. Suppose $f \in C_B^2[0,\infty)$ with respect to x applying Taylor's theorem, we receive

$$\psi(s) = \psi(x) + (s-x)\psi'(x) + \frac{\psi''(\eta)}{2}(s-x)^2, \quad \eta \in (x,s).$$

This results in

$$L_m^{*(\sigma_1,\sigma_2)}(\psi;x) - \psi(x) = \psi'(x)M_{1,m}^{(\sigma_1,\sigma_2)}(x) + \frac{\psi''(\eta)}{2}M_{2,m}^{(\sigma_1,\sigma_2)}(x).$$

From this, it follows easily that.

$$\left|L_m^{*(\sigma_1,\sigma_2)}(\psi;x) - \psi(x)\right| \leq \left|\psi'(x)M_{1,m}^{(\sigma_1,\sigma_2)}(x) + \frac{\psi''(\eta)}{2}M_{2,m}^{(\sigma_1,\sigma_2)}(x)\right|$$

$$\leq \left[M_{1,m}^{(\sigma_1,\sigma_2)}(x) + M_{2,m}^{(\sigma_1,\sigma_2)}(x)\right]||\psi||_{C_B[0,\infty)}^2. \tag{3.2}$$

With Lemma 2.2. and (2.1) equality, the estimation can be written as

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - f(x)\right| = \left|L_m^{*(\sigma_1,\sigma_2)}(f;x) - L_m^{*(\sigma,\sigma_2)}(\psi;x) + L_m^{*(\sigma,\sigma_2)}(\psi;x) - \psi(x) - f(x) + \psi(x)\right|$$

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$$
\begin{aligned}
&\le \left|L_m^{*(\sigma_1,\sigma_2)}(f-\psi;x)\right| + \left|L_m^{*(\sigma_1,\sigma_2)}(\psi;x)-\psi(x)\right| + |f(x)-\psi(x)| \\
&\le 2\|f-\psi\|_{C_B[0,\infty)} + \left|L_m^{*(\sigma_1,\sigma_2)}(\psi;x)-\psi(x)\right| \\
&\le 2\|f-\psi\|_{C_B[0,\infty)} + 2\delta_m(x)\|\psi\|_{C_B^2[0,\infty)} \\
&= 2\left(\|f-\psi\|_{C_B[0,\infty)} + \delta_m(x)\|\psi\|_{C_B^2[0,\infty)}\right).
\end{aligned}
$$

By finding the infimum among all $\psi \in C_B^2[0,\infty)$ and combining it with the definition of $K(f;.)$, we can deduce the intended result from the last inequality

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x)-f(x)\right| \le 2K\big(f;\lambda_m(x)\big).$$

Theorem 3.4. Suppose $f \in C_B[0,\infty)$. Then

$$\left|L_m^{*(\sigma_1,\sigma_2)}(f;x)-f(x)\right| \le C\omega_2\left(f;\sqrt{\mu_m(x)}\right) + \omega\left(f;M_{1,m}^{(\sigma_1,\sigma_2)}(x)\right)$$

here C is a positive constant and

$$\mu_m(x) = \frac{1}{8}\left\{M_{1,m}^{(\sigma_1,\sigma_2)}(x) + \left[M_{2,m}^{(\sigma_1,\sigma_2)}(x)\right]^2\right\},$$

and $\omega_2(f;.)$ denote the second order modulus of smoothness of function f .

Proof. Here we take the operators $F_m^{(\sigma_1,\sigma_2)}$ given by

$$F_m^{(\sigma_1,\sigma_2)}(f;x) = L_m^{*(\sigma_1,\sigma_2)}(f;x) - f\left(L_m^{*(\sigma_1,\sigma_2)}(s;x)\right) + f(x). \tag{3.3}$$

We can also derive this From Lemma 2.2.

$$F_m^{(\sigma_1,\sigma_2)}(s-x;x) = 0. \tag{3.4}$$

Furthermore, $\psi \in C_B^2[0,\infty)$ the subsequent equality can be derived using the Taylor formula,

$$\psi(s) = \psi(x) + (s-x)\psi'(x) + \int_x^s (s-u)\,\psi''(u)du\,.$$

If we use the operator which is defined by (3.3), then

$$
\begin{aligned}
\left|F_m^{(\sigma_1,\sigma_2)}(\psi;x)-\psi(x)\right| &= \left|F_m^{(\sigma_1,\sigma_2)}\left(\int_x^s (s-u)\,\psi''(u)du;x\right)\right| \\
&= \Bigg|L_m^{*(\sigma_1,\sigma_2)}\left(\int_x^s (s-u)\,\psi''(u)du;x\right) \\
&\quad - \int_x^{L_m^{*(\sigma_1,\sigma_2)}(s;x)} \left(L_m^{*(\sigma_1,\sigma_2)}(s;x)-u\right)\psi''(u)du + \int_x^x (s-u)\,\psi''(u)du\Bigg| \\
&\le \left|L_m^{*(\sigma_1,\sigma_2)}\left(\int_x^s (s-u)\,\psi''(u)du;x\right)\right| \\
&\quad + \left|\int_x^{L_m^{*(\sigma_1,\sigma_2)}(s;x)} \left(L_m^{*(\sigma_1,\sigma_2)}(s;x)-u\right)\psi''(u)du\right|.
\end{aligned}
$$

If we use the relation which is defined by (2.1), then

$$\begin{aligned}\left|F_m^{(\sigma_1,\sigma_2)}(\psi;x)-\psi(x)\right| &\le \frac{1}{2}\left\{M_{2,m}^{(\sigma_1,\sigma_2)}(x)+\left[M_{1,m}^{(\sigma_1,\sigma_2)}(x)\right]^2\right\}\|\psi''\|_{C_B[0,\infty)}\\ &\le 4\mu_m(x)\|\psi\|_{C_B^2[0,\infty)}. \end{aligned} \tag{3.5}$$

Combining the definition of the operator $F_n^{(\sigma_1,\sigma_2)}$, Lemma 2.2 and (3.5), we acquire the estimation

$$\begin{aligned}\left|L_m^{*(\sigma_1,\sigma_2)}(f;x)-f(x)\right| &= \Big|F_m^{(\sigma_1,\sigma_2)}(f-\psi;x)-(f-\psi)(x)+F_m^{(\sigma_1,\sigma_2)}(\psi;x)-\psi(x)\\ &\quad +f\left(L_m^{*(\sigma_1,\sigma_2)}(s;x)\right)-f(x)\Big|.\end{aligned}$$

Using the triangle inequality

$$\begin{aligned}\left|L_m^{*(\sigma_1,\sigma_2)}(f;x)-f(x)\right| &\le \left|F_m^{(\sigma_1,\sigma_2)}(f-\psi;x)-(f-\psi)(x)\right|+\left|F_m^{(\sigma_1,\sigma_2)}(\psi;x)-\psi(x)\right|\\ &\quad +\left|f\left(L_m^{*(\sigma_1,\sigma_2)}(s;x)\right)-f(x)\right|\\ &\le 4\|f-\psi\|_{C_B[0,\infty)}+4\mu_m(x)\|\psi\|_{C_B^2[0,\infty)}+\omega\left(f;L_m^{*(\sigma_1,\sigma_2)}(s-x;x)\right)\\ &\le 4\|f-\psi\|_{C_B[0,\infty)}+4\mu_m(x)\|\psi\|_{C_B^2[0,\infty)}+\omega\left(f;M_{1,m}^{(\sigma_1,\sigma_2)}(x)\right).\end{aligned}$$

Considering the relation between $K(f;.)$ and $\omega_2(f;.)$ which is given by (3.1), then we have

$$\begin{aligned}\left|L_m^{*(\sigma_1,\sigma_2)}(f;x)-f(x)\right| &\le 4K\big(f;\mu_m(x)\big)+\omega\left(f;M_{1,m}^{(\sigma_1,\sigma_2)}(x)\right)\\ &\le C\omega_2\left(f;\sqrt{\mu_m(x)}\right)+\omega\left(f;M_{1,m}^{(\sigma_1,\sigma_2)}(x)\right).\end{aligned}$$

4. CONCLUSION

In this paper, a brief introduction is provided for the Szász operator. This operator serves as the foundation for exploring various approximation properties, leading to the creation of additional operators and polynomials. One such operator is the Cheney and Sharma (1964) operator, giving rise to the emergence of orthogonal polynomials. Subsequently, a model involving Appell polynomials is devised by Jakimovski and Leviatan (1969). Following this, a study focusing on Brenke polynomials is conducted by Varma et al. (2012). Additionally, a Kantorovich operator containing Brenke polynomials is introduced, along with an exploration of the operator which is defined by Waterman incorporating generalized Brenke polynomials by Sucu (2022). These operators, with their distinct formulations, are presented in a different format, specifically expressed as (1.9) and encompassing an expansion of Brenke polynomials.

5. RESULT

In this paper, we have generalized the operators that are defined by Sucu (2022), and we defined them in different shapes, it is shown as (1.9), where we call them Stancu Kantorovich type operators, including generalized Brenke polynomials. Also, the approximation of functions is discussed. As mentioned in the introduction section, with the motivation of these newly defined operators, we believe that it will be helpful and valuable for the readers.

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