Adıyaman University Journal of Science
https://dergipark.org.tr/en/pub/adyujsci
DergiPark AKADEMİK
ISSN 2147-1630
e-ISSN 2146-586X

On the Interval Valued Cesaro Convergent Sequences Space

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Received: 14.05.2023 **Accepted:** 06.07.2023 **Published:** 31.12.2023

Abstract

The concept of quasilinear space is a field that needs to be matured, the foundations of which were laid by S. M. Aseev's published work in 1986. The simplest nonlinear quasi linear space example is the set P which is a class of closed intervals of real numbers. In this study, it was given an interval-valued sequence space using the Cesàro limitation method's matrix domain. Also, its quasilinear space structure, some topological characteristics, and some inclusion relations were examined.

Keywords: Quasilinear Space; Interval Valued Sequence; Hausdorff Metric; Cesàro Convergence.

Aralık Değerli Cesaro Yakınsak Diziler Uzayı Üzerine

Öz

Quasilineer uzay kavramı, temelleri S. M. Aseev'in 1986 yılında yayınlanan çalışmasıyla atılan, olgunlaşması gereken bir alandır. Lineer olmayan Quasilineer uzayın en basit örneği, gerçek sayıların kapalı aralıklar sınıfı olan P kümesidir. Bu çalışmada Cesàro limitleme yönteminin matris etki alanı kullanılarak aralık değerli bir dizi uzayı verildi. Ayrıca bu uzayın quasilineer uzay yapısı, bazı topolojik özellikleri ve bazı kapsama ilişkileri incelendi.

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DOI: 10.37094/adyujsci.1296930

Anahtar Kelimeler: Quasilineer Uzay; Aralık Değerli Dizi; Hausdorff Metrik; Cesàro yakınsaklık.

1. Introduction

Aseev [13] introduced the concept of quasilinear spaces in 1986, which generalized linear spaces. The partial order relation he used in the definition made it easy to give a consistent response to some basic concepts and results of linear algebra. His study has also inspired the presentation of many studies on set valued analysis [14], set differential equations [15], fuzzy quasilinear spaces [16]. Yılmaz has many studies on quasilinear spaces and has made important contributions to the literature [17, 18, 24-31].

After Zadeh [1] introduced the concept of fuzzy set to the literature, interval numbers and fuzzy numbers have also been used in the construction of mathematical structures. One of these structures is sequence spaces. Interval arithmetic, which was founded by Dwyer [2], was further developed by Moore [3, 4]. Chiao introduced sequence of interval numbers and defined the usual convergence of sequence interval numbers [23]. Some other studies on this topic: [12], [13].

Some studies in which fuzzy sequence spaces are defined and some of their properties are examined [6-11, 20]:

In this paper, we introduce interval valued Cesàro convergent sequence spaces and discuss some of their properties.

2. Preliminaries

Throughout this study, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will represent the set of natural, real and complex numbers, respectively. Closed interval is a subset of the real numbers $\{x \in \mathbb{R} : \underline{D} \leq x \leq \underline{D}\}$ and the interval D is denoted as $D = [\underline{D}, \overline{D}]$, where $\underline{D}$ and $\overline{D}$ are the left and right endpoints of an interval D, respectively [5]. Although other types of intervals (open, half-open) appear in mathematics, our focus will be on closed intervals. The interval term in this study refers to a closed interval.

D is said to be degenerate, if $\underline{D} = \overline{D}$. An interval of this type contains a single real number d, [5].

A closed subset of real numbers is an interval number. It is denoted as a set of all real valued interval numbers by P in this study. That is, each element of P is D, represented as

$$D = \left\{x \in \mathbb{R} : \underline{D} \leq x \leq \overline{D}\right\}. \tag{1}$$

The set of all real valued interval numbers P is metric space with the metric h called the hausdorff metric, [5] where h is defined as

$$h(D_1, D_2) = max\left\{\left|\underline{D_1} - \underline{D_2}\right|, \left|\overline{D_1} - \overline{D_2}\right|\right\}. \tag{2}$$

It is easily obtained that P is complete metric space with the function h. The usual metric of $\mathbb{R}$ is obtained when D_1 and D_2 are degenerate intervals.

Also, the set of all real-valued interval numbers P is the normed space with the norm function defined [5]:

$$\|D\|_P = sup\|t\|_{\mathbb{R}}, \qquad t \in D, \qquad D \in P, \qquad \|t\|_{\mathbb{R}} = |t|.$$

Let us now go over Aseev's [13] definition of a quasilinear space and some of its basic properties.

The operations of addition, scalar multiplication, and a partial order relation on the set P of intervals are defined as follows:

For all $U, V \in P$, $U = \left[\underline{U}, \overline{U}\,\right]$, $V = \left[\underline{V}, \overline{V}\right]$ and $\lambda \in \mathbb{R}$,

$$U + V = \left[\underline{U}, \overline{U}\,\right] + \left[\underline{V}, \overline{V}\right] = [\underline{U} + \underline{V}, \overline{U} + \overline{V}]$$

$$\lambda U = \lambda\left[\underline{U}, \overline{U}\right] = \begin{cases} \left[\lambda\underline{U}\,, \lambda\overline{U}\right], & if\ \ \lambda \geq 0, \\ \left[\lambda\overline{U}, \lambda\underline{U}\right], & if\ \lambda < 0, \end{cases}$$

$$U \leq V \Leftrightarrow \left[\underline{U}, \overline{U}\,\right] \subseteq \left[\underline{V}, \overline{V}\right].$$

Let us continue by defining quasilinear space.

Definition 1: [13] When the addition, scalar multiplication, and partial order relation defined on a set X satisfy the following conditions, X is called quasilinear space:

i) $u \preccurlyeq u$,

ii) $u \preccurlyeq w$, if $u \preccurlyeq v$ and $v \preccurlyeq w$,

iii) $u = v$ if $u \preccurlyeq v$ and $v \preccurlyeq u$,

iv) $u + v = v + u$,

v) $u + (v + w) = (v + u) + w$,

vi) There is a θ element of X that satisfies $x + \theta = x$,

vii) $\mu.(\lambda u) = (\mu.\lambda)u$,

viii) $\mu.(u+v)=\mu.u+\mu.v$,

ix) $1.u=u$,

x) $0.u=\theta$,

xi) $(\mu+\lambda).u \preccurlyeq \mu.u+\lambda.v$,

xii) $u+v \preccurlyeq w+\varphi$ if $u \preccurlyeq w$ and $v \preccurlyeq \varphi$,

xiii) $\mu.u \preccurlyeq \mu.v$ if $u \preccurlyeq v$,

where for all $u, v, w, \varphi \in X$ and $\mu, \lambda \in \mathbb{R}$.

In this study, quasilinear space is abbreviated as QLS.

When we use $'' = ''$ as a partial order relation, then the QLS transforms into a linear space. The set of all closed intervals of real numbers is a famous understandable example of a non-linear QLS.

Definition 2: [13] A norm is a n function defined from an X QLS to $\mathbb{R}$ that satisfies the following conditions, in this case X is called a normed quasilinear space. That is $n: X \to \mathbb{R}$,

For all $u, v \in X$ and μ scalar,

i. $n(u) > 0$, if $u \neq \theta$

ii. $n(u+v) \leq n(u)+n(v)$,

iii. $n(\mu.u) = |\mu| n(u)$,

iv. $n(u) \leq n(v)$, if $u \preccurlyeq v$,

v. For any positive number δ, If X has an u_δ element that satisfies $n(u_\delta) \leq \delta$ and, $u \leq v + u_\delta$ then $u \leq v$.

Now let us define definition of the concept of interval number sequence.

An interval number sequence is a function whose domain set is ℕ, and the range set is the set of closed intervals P. That is, the function f defined as follows:

$$f: N \to P, \qquad f(k) = (D_k),$$

is called an interval number sequence, where $D_k = [\underline{D_k}, \overline{D}_k]$, for each k, and $\underline{D_k} \leq \overline{D}_k$, [12].

The class of all interval sequences will be denoted by $w(P)$ in this study.

$$w(P) = \{(D_k): k \in N, \quad D_k \in P\},$$

The space $w(P)$ is a quasilinear space with the following operations defined on interval term sequences: For all $U, V \in w(P)$,

$$U = (U_k) = ([\underline{U}_1, \overline{U}_1], [\underline{U}_2, \overline{U}_2], \dots, [\underline{U}_k, \overline{U}_k], \dots)$$

$$V = (V_k) = ([\underline{V}_1, \overline{V}_1], [\underline{V}_2, \overline{V}_2], \dots, [\underline{V}_k, \overline{V}_k], \dots)$$

$$U + V = ([\underline{U}_k + \underline{V}_k, \overline{U}_k + \overline{V}_k]) \tag{3}$$

$$\lambda U = \lambda([\underline{U}_k, \overline{U}_k]) = (\lambda[\underline{U}_k, \overline{U}_k])$$

$$\lambda[\underline{U}_k, \overline{U}_k] = \begin{cases} [\lambda\underline{U}, \lambda\overline{U}], & \lambda \geq 0 \\ [\lambda\overline{U}, \lambda\underline{U}], & \lambda < 0 \end{cases} \tag{4}$$

$$U \preccurlyeq V \Leftrightarrow [\underline{U}_k, \overline{U}_k] \subseteq [\underline{V}_k, \overline{V}_k]. \tag{5}$$

The following is a definition of the convergence of interval number sequences, [23]:

Let (U_k) be a sequence of interval numbers and U_0 be an interval number. If there is a $k_0 = k_0(\varepsilon) \in N$ for which the inequality $h(U_k, U_0) < \varepsilon$ is provided for all $\varepsilon > 0$ and for all $k > k_0$, then the sequence (U_k) is said to be convergent to U_0. This convergence is displayed as

$$\lim_k U_k = U_0 \text{ or } (U_k) \to U_0, \ (k \to \infty),$$

where the limit is taken on the Haussdorff metric h given by equation 2.

We can conclude that $\lim_{k\to\infty} U_k = U_0 \Leftrightarrow \lim_{k\to\infty} \underline{U}_k = \underline{U}_0$ ve $\lim_{k\to\infty} \overline{U}_k = \overline{U}_0$.

For example, let us consider the interval number sequence $(U_k) = [-\frac{1}{k+3}, \frac{1}{k+3}]$. If we examine the convergence of this sequence of intervals, we get that

$\lim_{k\to\infty} U_k = \lim_{k\to\infty} \left[-\frac{1}{k+3}, \frac{1}{k+3}\right] = [0,0] = \theta$. This means that the sequence (U_k) is convergent to the interval number $[0,0] = \theta$.

The spaces of null, convergent, and bounded sequences of interval numbers are defined as follows, respectively:

$$P_{c_0} = \left\{(U_k) \in w(P): \lim_k U_k = [0,0] = \theta\right\},$$

$$P_c = \left\{(U_k) \in w(P): \lim_k U_k = U_0, \qquad U_0 \in P\right\},$$

$$P_{\ell_\infty} = \left\{(U_k) \in w(P): \sup_k\{|\underline{U}_k|, |\overline{U}_k|\} < \infty\right\}.$$

These spaces are complete metric spaces with the function d defined as follows for each U and V sequence taken from these spaces: [12].

$$d(U_k, V_k) = \sup_k\{\max\{|\underline{U}_k - \underline{V}_k|, |\overline{U}_k - \overline{V}_k|\}\}.$$

An interval valued sequence is Cesàro convergent to $V \in P$ if and only if $h(\frac{1}{n}\sum_{k=1}^{n} U_k, V) \to 0$ for $n \to \infty$.

3. Main Results

$C_I(1)$ and $C_{I_0}(1)$ represent the spaces of interval valued Cesàro convergent sequences and interval valued Cesàro null convergent sequences, respectively, in this study. That is,

$$C_I(1) = \{(U_k) \in w(P) : \lim_{n\to\infty} h\left(\tfrac{1}{n}\textstyle\sum_{k=1}^{n}[\underline{U}_k, \overline{U}_k], [\underline{V}, \overline{V}]\right) = 0, \text{ for some } [\underline{V}, \overline{V}] \in P\}$$

$$C_{I_0}(1) = \{(U_k) \in w(P) : \lim_{n\to\infty} h\left(\frac{1}{n}\sum_{k=1}^{n}[\underline{U}_k, \overline{U}_k], [0,0]\right) = 0\}$$

Now let us show that this set is well defined, that is, it has at least one element.

For instance, the sequence $([-1, \frac{1}{n}])$ belongs to the set $C_I(1)$. Really it can be easily seen that

$$\lim_n \frac{1}{n}\sum_{i=1}^{n}\left[-1, \frac{1}{i}\right] = [-1,0]$$

Therefore $C_I(1) \neq \emptyset$.

Theorem 1: The space $C_I(1)$ is a metric space with the function d defined as

$$d(U, V) = \sup_n\{\max\left\{\tfrac{1}{n}\textstyle\sum_{i=1}^{n}|\underline{U}_i - \underline{V}_i|, \tfrac{1}{n}\textstyle\sum_{i=1}^{n}|\overline{U}_i - \overline{V}_i|\right\}\}. \tag{6}$$

Proof: i) For all $U, V \in C_I(1)$, when $U \neq V$, it is easy to see that $d(U, V) > 0$,

Let us prove that $U = V \Leftrightarrow d(U, V) = 0$.

$U = V \Leftrightarrow U_i = V_i$, for each $i \in \mathbb{N}$,

$\Leftrightarrow \underline{U}_i = \underline{V}_i$ and $\overline{U}_i = \overline{V}_i$, for each $i \in \mathbb{N}$, From here

$\left|\underline{U}_i - \underline{V}_i\right|$ =0, and $\left|\overline{U_i} - \overline{V}_i\right| = 0 \Leftrightarrow d(U,V) = 0$

ii) It is clear that $d(U,V) = d(V,U)$

iii) $d(U,V) = \sup_n \left\{ \max\left\{ \frac{1}{n}\sum_{i=1}^{n}\left|\underline{U}_i - \underline{V}_i\right|, \frac{1}{n}\sum_{i=1}^{n}\left|\overline{U_i} - \overline{V}_i\right| \right\} \right\}$

$$=\sup_n \left\{ \max\left\{ \frac{1}{n}\sum_{i=1}^{n}\left|\underline{U}_i - \underline{V}_i - \underline{Z}_i + \underline{Z}_i\right|, \frac{1}{n}\sum_{i=1}^{n}\left|\overline{U_i} - \overline{V}_i - \overline{Z}_i + \overline{Z}_i\right| \right\} \right\}$$

$$\leq \sup_n \left\{ \max\left\{ \frac{1}{n}\sum_{i=1}^{n}\left(\left|\underline{U}_i - \underline{Z}_i\right| + \left|\underline{Z}_i - \underline{V}_i\right|\right), \frac{1}{n}\sum_{i=1}^{n}\left(\left|\overline{U_i} - \overline{Z}_i\right| + \left|\overline{Z}_i - \overline{V}_i\right|\right) \right\} \right\}$$

$$=\sup_n \left\{ \max\left\{ \frac{1}{n}\sum_{i=1}^{n}\left(\left|\underline{U}_i - \underline{Z}_i\right|, \left|\overline{U_i} - \overline{Z}_i\right|\right) + \frac{1}{n}\sum_{i=1}^{n}\left(\left|\underline{Z}_i - \underline{V}_i\right|, \left|\overline{Z}_i - \overline{V}_i\right|\right) \right\} \right\}$$

$$=\sup_n \left\{ \max\left\{ \frac{1}{n}\sum_{i=1}^{n}\left(\left|\underline{U}_i - \underline{Z}_i\right|, \left|\overline{U_i} - \overline{Z}_i\right|\right) \right\} + \max\left\{ \frac{1}{n}\sum_{i=1}^{n}\left(\left|\underline{Z}_i - \underline{V}_i\right|, \left|\overline{Z}_i - \overline{V}_i\right|\right) \right\} \right\}$$

$$\leq \sup_n \left\{ \max\left\{ \frac{1}{n}\sum_{i=1}^{n}\left(\left|\underline{U}_i - \underline{Z}_i\right|, \left|\overline{U_i} - \overline{Z}_i\right|\right) \right\} \right\} + \sup_n \left\{ \max\left\{ \frac{1}{n}\sum_{i=1}^{n}\left(\left|\underline{Z}_i - \underline{V}_i\right|, \left|\overline{Z}_i - \overline{V}_i\right|\right) \right\} \right\}$$
$$= d(U,Z) + d(Z,V)$$

Thus, it can be written that

$d(U,V) \leq d(U,Z) + d(Z,V)$.

That is, d is a metric.

Theorem 2: The spaces $(C_I(1), d), \left(C_{I_0}(1), d\right)$ are complete metric spaces with the metric d defined in (6).

Proof: Consider (U^m) as a Cauchy sequence in $C_I(1)$ space. We get

$$d(U^r, U^m) \to 0, (r, m \to \infty),$$

where $U^r = (U_k^r) = (U_1^r, U_2^r, \dots, U_k^r, \dots)$, $U^m = (U_k^m)$ and $U_k^r = \left[\underline{U}_k{}^r, \overline{U}_k^r\right]$. From here, for $(r, m \to \infty)$

$$\sup_n \left\{ \max\left\{ \frac{1}{n}\sum_{k=1}^{n}\left|\underline{U}_k{}^r - \underline{U}_k{}^m\right|, \frac{1}{n}\sum_{k=1}^{n}\left|\overline{U}_k^r - \overline{U}_k^m\right| \right\} \right\} \to 0.$$

It means that for each n,

$$\max\left\{ \frac{1}{n}\sum_{k=1}^{n}\left|\underline{U}_k{}^r - \underline{U}_k{}^m\right|, \frac{1}{n}\sum_{k=1}^{n}\left|\overline{U}_k^r - \overline{U}_k^m\right| \right\} \to 0, \quad (r, m \to \infty).$$

Hence, for each n,

$$\frac{1}{n}\sum_{k=1}^{n}\left|\underline{U}_k{}^{r}-\underline{U}_k{}^{m}\right|\to 0, \text{ and } \frac{1}{n}\sum_{k=1}^{n}\left|\overline{U}_k^{r}-\overline{U}_k^{m}\right|\to 0.$$

This shows us that cesaro transform of the sequences of real numbers $\left(\underline{U}_k{}^{m}\right)_{m=1}^{\infty}$ and $\left(\overline{U}_k^{m}\right)_{m=1}^{\infty}$ is the cauchy sequence. If the space of Cesàro convergent classical real number sequences is denoted by $c(1)$, we know that this sequence space is complete. Therefore, suppose that the sequences $\left(\underline{U}_k{}^{m}\right)$ and $\left(\overline{U}_k^{m}\right)$ are Cesàro convergent to the numbers $\underline{U}_k$ and $\overline{U}_k$, respectively. Since we know that for all k, $\underline{U}_k{}^{m}\leq\overline{U}_k^{m}$, we write $\underline{U}_k\leq\overline{U}_k$. Take into account the interval $U=\left[\underline{U}_k,\overline{U}_k\right]$ thus produced. Then, we get $d(U^r,U^m)\to d(U,U^m)$, for $r\to\infty$.

Because the sequences $\left(\underline{U}_k{}^{m}\right)$ and $\left(\overline{U}_k^{m}\right)$ are Cesàro convergent to the numbers $\underline{U}_k$ and $\overline{U}_k$, respectively, we can say that for all positive numbers ε_1 and ε_2, there are the numbers $m_0(\varepsilon_1)$ and $m_0(\varepsilon_2)$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\left|\underline{U}_k-\underline{U}_k{}^{m}\right|\leq\varepsilon_1$$

when $m>m_0(\varepsilon_1)$ and

$$\frac{1}{n}\sum_{k=1}^{n}\left|\overline{U}_k-\overline{U}_k^{m}\right|\leq\varepsilon_2,$$

when $m>m_0(\varepsilon_2)$.

Let us say ε to the maximum of ε_1 and ε_2 numbers, that is $\varepsilon=max\{\varepsilon_1,\varepsilon_2\}$. In this case, for

$$m>max\{m_0(\varepsilon_1),m_0(\varepsilon_2)\},$$

$$max\left\{\frac{1}{n}\sum_{k=1}^{n}\left|\underline{U}_k-\underline{U}_k{}^{m}\right|,\frac{1}{n}\sum_{k=1}^{n}\left|\overline{U}_k-\overline{U}_k^{m}\right|\right\}<\varepsilon. \tag{7}$$

If the course of the proof is followed, it is seen that the (7) inequality is provided for each n. It means that

$$d(U,U^m)=\sup_n\left\{max\left\{\frac{1}{n}\left|\underline{U}_k-\underline{U}_k{}^{m}\right|,\frac{1}{n}\sum_{k=1}^{n}\left|\overline{U}_k-\overline{U}_k^{m}\right|\right\}\right\}\leq\varepsilon.$$

This indicates that it is $U\in C_I(1)$. Thus, it has completed the proof.

Theorem 3: The spaces P_c and $C_I(1)$ are not linear isomorphic.

Proof: To demonstrate that two spaces are linear isomorphic, we must demonstrate the existence of a one-to-one and onto a linear transformation between them.

Let us define the transform $T, T: P_c \to C_I^{(1)}$,

$$U \to T(U) = V, \quad V = (V_n), \qquad (V_n) = \frac{1}{n}\sum_{k=1}^{n}\left[\underline{U}_k, \overline{U}_k\right] \qquad (n \in \mathbb{N}).$$

For all $X = (X_k) = \left[\underline{X}_k, \overline{X}_k\right]$ and $U = (U_k) = \left[\underline{U}_k, \overline{U}_k\right] \in P_c$,

$$T(X+U) = \frac{1}{n}\sum_{k=1}^{n}\left[\underline{X}_k + \underline{U}_k, \overline{X}_k + \overline{U}_k\right]$$

$$= T(X) + T(U)$$

$$T(\alpha.X) = \tfrac{1}{n}\textstyle\sum_{k=1}^{n}[\alpha\,\underline{X}_k, \alpha\overline{X}_k] = \alpha.(\tfrac{1}{n}\textstyle\sum_{k=1}^{n}[\,\underline{X}_k, \overline{X}_k]) = \alpha.T(X), \ (\alpha \in \mathbb{R}).$$

So, the transform T is linear.

Let us investigate the T transform's one-to-one. For this purpose, let us assume

$T(X) = T(U)$. Then, we get $T(X) - T(U) = \theta$

where $\theta = [0,0]$. $T(X-U) = \theta$ is obtained, because the T transform is linear.

$$T(X-U) = \frac{1}{n}\sum_{k=1}^{n}([\,\underline{X}_k, \overline{X}_k] - \left[\underline{U}_k, \overline{U}_k\right]) = [0,0],$$

$$= \tfrac{1}{n}\textstyle\sum_{k=1}^{n}(\left[\underline{X}_k - \overline{U}_k, \overline{X}_k - \underline{U}_k\right] = [0,0].$$

This requires $\underline{X}_k = \overline{U}_k$, and $\overline{X}_k = \underline{U}_k$ for all $k \in \mathbb{N}$. This means $\left[\,\underline{X}_k, \overline{X}_k\right] \neq \left[\underline{U}_k, \overline{U}_k\right]$.

So, T is not $1{:}1$. Therefore, the spaces P_c and $C_I(1)$ are not linear isomorphic.

Theorem 4: An interval valued convergent sequence is also Cesáro convergent.

Proof: Let us assume that the sequence $X = (X_n)$ converges to an element $Y = [\,\underline{Y}, \overline{Y}]$ in P. That is $\lim\limits_{n\to\infty} h\,(X_n, \mathrm{Y}) = 0$. We get from here that,

$$\lim_{n\to\infty} max\{\left|\underline{X}_n - \underline{Y}\right|, \left|\,\overline{X}_n - \overline{Y}\right|\}\text{=0}$$

This means that $\lim_{n\to\infty}\left|\underline{X}_n - \underline{Y}\right| = 0$ ve $\lim_{n\to\infty}\left|\overline{X}_n - \overline{Y}\right| = 0$. Thus, the real sequences$\left(\underline{X}_n\right)$ and $(\overline{X}_n)$ are convergent. We know that sequences of convergent real numbers are Cesáro convergent.

Let us assume that the sequences $\left(\underline{X}_n\right)$ and $(\overline{X}_n)$ of real numbers Cesàro converge to $\underline{U}$ and $\overline{U}$. Where $\underline{U}$ and $\overline{U}$ are real numbers. Since for all $n \in \mathbb{N}$, $\underline{X}_n \leq \overline{X}_n$, we know that $\lim_{n\to\infty}\underline{X}_n \leq \lim_{n\to\infty}\overline{X}_n$. From here, we have $\underline{U} \leq \overline{U}$. In addition, since the Cesáro limit and the limit of a convergent sequence in sequences with real terms are equal,

$$\lim_{n\to\infty} h\left(\left[\underline{X}_n, \overline{X}_n\right], \left[\underline{U}, \overline{U}\right]\right) = \lim_{n\to\infty}\left(max\left\{\left|\underline{X}_n - \underline{U}\right|, \left|\overline{X}_n - \overline{U}\right|\right\}\right)$$

$$= \lim_{n\to\infty}\left(max\left\{\left|\underline{X}_n - \underline{Y}\right|, \left|\overline{X}_n - \overline{Y}\right|\right\} \quad (n \to \infty).$$

It means that the sequence $X = (X_n)$ Cesàro converges to $Y = [\,\underline{Y}, \overline{Y}]$.

Consequently, we can say that the sequence space $C_I(1)$ contains the sequence space P_c. That is, $P_c \subset C_I(1)$. Similarly, $P_{c_0} \subset C_{I_0}(1)$ inclusion relation is also valid.

Example 1: Consider the following sequence $(X_n) = \left([1 + \frac{1}{n}, 3]\right)$. We see that it converges to [1,3]. We can prove easily. Really,

$$h\left(\left[1 + \tfrac{1}{n}, 3\right], [1,3]\right) = max\left\{\left|1 + \tfrac{1}{n} - 1\right|, |3 - 3|\right\}$$

$$= max\left\{\tfrac{1}{n}, 0\right\} = \tfrac{1}{n} \to 0, (n \to \infty).$$

Now, let us investigate Cesàro convergence. We must demonstrate that

$$h\left(\frac{1}{n}\sum_{k=1}^{n} X_n\ , [1,3]\right) \to 0, \quad (n \to \infty).$$

$$h(\frac{1}{n}\sum_{k=1}^{n} X_n\ , [1,3]) = h(\frac{1}{n}\sum_{k=1}^{n}\ [\,1 + \frac{1}{k}\ ,3], [1,3])$$

$$= h(\frac{1}{n}([1 + 1, 3] + [1 + \frac{1}{2}, 3] + \cdots + [1 + \frac{1}{n}, 3]), [1,3])$$

$$= h((\left[\frac{1}{n} + \frac{1}{n}, \frac{3}{n}\right] + \left[\frac{1}{n} + \frac{1}{2n}, \frac{3}{n}\right] + \cdots + \left[\frac{1}{n} + \frac{1}{n.n}, \frac{3}{n}\right]), [1,3])$$

$$= h([1 + \frac{1}{n}(1 + \frac{1}{2} + \cdots + \frac{1}{n}), 3], [1,3])$$

$$= max\left\{\left|1 + \frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - 1\right|, |3 - 3|\right\}$$

$$= \frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \to 0, (n \to \infty)$$

So, the proof is completed.

Now, let us see with an example that an interval sequence that is divergent can be convergent according to the Hausdorff metric in the Cesáro sense.

Example 2: Although the sequence $X = (X_n) = ([-1,0], [0,1], [-1,0], [0,1], \dots)$ is divergent, it is Cesàro convergent to $\left[-\frac{1}{2}, \frac{1}{2}\right]$ according to the Hausdorff metric. Let us examine the Cesàro convergence depending on n is even or odd which stamps the terms of the sequence.

If n is an odd number, then $X_{2n+1} = \left[-\frac{n+1}{2n+1}, \frac{n}{2n+1}\right]$.

Let us try to demonstrate that $\lim_{n\to\infty} h(X_{2n+1}, [-\frac{1}{2}, \frac{1}{2}]) = 0$.

$$\lim_{n\to\infty} h(X_{2n+1}, [-\frac{1}{2}, \frac{1}{2}]) = \lim_{n\to\infty} h(\left[-\frac{n+1}{2n+1}, \frac{n}{2n+1}\right], [-\frac{1}{2}, \frac{1}{2}])$$

$$= \lim_{n\to\infty}(max\{\left|-\frac{n+1}{2n+1} - (-\frac{1}{2})\right|, \left|\frac{n}{2n+1} - \frac{1}{2}\right|\})$$

$$= \lim_{n\to\infty}(max\left\{\left|-\frac{n+1}{2n+1} + \frac{1}{2})\right|, \left|\frac{n}{2n+1} - \frac{1}{2}\right|\right\})$$

If here the maximum expression is $\left|-\frac{n+1}{2n+1} + \frac{1}{2}\right|$, we get $\lim_{n\to\infty}\left|-\frac{n+1}{2n+1} + \frac{1}{2}\right| = 0$.

If here the maximum expression is $\left|\frac{n}{2n+1} - \frac{1}{2}\right|$, then $\lim_{n\to\infty}\left|\frac{n}{2n+1} - \frac{1}{2}\right| = 0$.

Let us now consider, the case where n is an even number. In this case, since

$X_{2n} = \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$\lim_{n\to\infty} h\left(X_{2n}, \left[-\frac{1}{2}, \frac{1}{2}\right]\right) = \lim_{n\to\infty} max\left\{\left|-\frac{1}{2} + \frac{1}{2}\right|, \left|\frac{1}{2} - \frac{1}{2}\right|\right\} = \lim_{n\to\infty} max\{0,0\} = 0.$$

As a result, the goal is accomplished.

Thus, we gave an example of an interval sequence that is Cesàro convergent even though it is divergent. This indicates that the coverage of $P_c \subset C_I(1)$ is certain.

Theorem 5: The space $C_I(1)$ is a subspace of P_{ℓ_∞}.

Proof: We will show that when $U = (U_k) \in C_I(1)$, U is the element of P_{ℓ_∞}.

$U \in C_I^{(1)} \Leftrightarrow \lim_{n\to\infty} h(\frac{1}{n}\sum_{k=1}^{n} U_k, V) = 0$, for any $V \in P$.

$$h\left(\frac{1}{n}\sum_{k=1}^{n}[\underline{U}_k, \overline{U}_k], V\right) = h\left(\frac{1}{n}([\underline{U}_1, \overline{U}_1] + \cdots + [\underline{U}_n, \overline{U}_n]), [\underline{V}, \overline{V}]\right),$$

$$= h\left(\left[\frac{\underline{U}_1 + \cdots + \underline{U}_n}{n}, \frac{\overline{U}_1 + \cdots + \overline{U}_n}{n}\right], [\underline{V}, \overline{V}]\right),$$

$$= max\left\{\left|\frac{1}{n}(\underline{U}_1 + \cdots + \underline{U}_n) - \underline{V}\right|, \left|\frac{1}{n}(\overline{U}_1 + \cdots + \overline{U}_n) - \overline{V}\right|\right\} \to 0, \qquad (n \to \infty)$$

$$\Leftrightarrow \left|\frac{1}{n}(\underline{U}_1 + \cdots + \underline{U}_n) - \underline{V}\right| \to 0 \text{ and } \left|\frac{1}{n}(\overline{U_1} + \cdots + \overline{U_n}) - \overline{V}\right| \to 0, \qquad (n \to \infty).$$

From here the sequences $(\underline{U}_k)$ and $(\overline{U}_k)$ are Cesàro convergent in $\mathbb{R}$. Therefore, they are also bounded. Since for all $k \in \mathbb{N}$, $\underline{U}_k \leq \overline{U}_k$, the interval $[\underline{U}_k, \overline{U}_k]$ is bounded and we get that $U = (U_k)$ belongs to P_{ℓ_∞}. We have, the space $C_I(1)$ is covered by P_{ℓ_∞}.

$$X \in C_I(1) \Leftrightarrow h(\left[\frac{1}{n}\left(\underline{X_1} + \cdots + \underline{X_n}\right), \frac{1}{n}\left(\overline{X_1} + \cdots + \overline{X_n}\right)\right], [\underline{U}, \overline{U}]) \to 0(n \to \infty),$$

$$Y \in C_I(1) \Leftrightarrow h\left(\left[\frac{1}{n}\left(\underline{Y_1} + \cdots + \underline{Y_n}\right), \frac{1}{n}\left(\overline{Y_1} + \cdots + \overline{Y_n}\right)\right], [\underline{V}, \overline{V}]\right) \to 0(n \to \infty),$$

where $U, V \in P$. Now, let us show that for $X, Y \in C_I(1)$, $X + Y \in C_I(1)$. Since

$$h(\left[\frac{1}{n}\left(\underline{X_1} + \underline{Y_1} + \cdots + \underline{X_n} + \underline{Y_n}\right), \left(\frac{1}{n}(\overline{X_1} + \overline{Y_1} + \cdots + \overline{X_n} + \overline{Y_n}\right)\right], [\underline{U} + \underline{V}, \overline{U} + \overline{V}]\Big)$$

$=h([\frac{1}{n}(\underline{X_1} + \cdots + \underline{X_n}), \frac{1}{n}(\overline{X_1} + \cdots + \overline{X_n})] + [\frac{1}{n}(\underline{Y_1} + \cdots + \underline{Y_n}), \frac{1}{n}(\overline{Y_1} + \cdots + \overline{Y_n})], \ [\underline{U}, \overline{U}] + [\underline{V}, V]) \to 0 \ (for\ n \to \infty)$, we get

$$h([\frac{1}{n}(\underline{X_1} + \underline{Y_1} + \cdots + \underline{X_n} + \underline{Y_n}), (\frac{1}{n}(\overline{X_1} + \overline{Y_1} + \cdots + \overline{X_n} + \overline{Y_n})], [\underline{U} + \underline{V}, \overline{U} + \overline{V}]) \to 0,$$

so $X + Y \in C_I(1)$.

For $X \in C_I(1)$ and $\lambda \in \mathbb{R}$, let us show that $\lambda X \in C_I(1)$

$$\lambda X = \lambda[\underline{X},\overline{X}] = \begin{cases}[\lambda\underline{X},\lambda\overline{X}], & if\ \lambda \geq 0\\ [\lambda\overline{X},\lambda\underline{X}], & if\ \lambda < 0\end{cases}$$

Since $X \in C_I(1) \Leftrightarrow h\left(\left[\frac{1}{n}\left(\underline{X_1} + \cdots + \underline{X_n}\right), \frac{1}{n}\left(\overline{X_1} + \cdots + \overline{X_n}\right)\right], [\underline{U},\overline{U}]\right) \to 0(n \to \infty)$,

$(U \in C_I(1))$. If $\lambda \geq 0$, $\lambda X = [\lambda\underline{X},\lambda\overline{X}]$. We have

$$h(\left[\frac{1}{n}\left(\lambda\underline{X_1} + \cdots + \lambda\underline{X_n}\right), \frac{1}{n}\left(\lambda\overline{X_1} + \cdots + \lambda\overline{X_n}\right)\right], [\lambda\underline{U},\lambda\overline{U}])$$

$$= h(\lambda[\frac{1}{n}(\underline{X_1} + \cdots + \underline{X_n}), \frac{1}{n}(\overline{X_1} + \cdots + \overline{X_n})], \lambda[\underline{U},\overline{U}])$$

$$= \lambda h\left(\left[\frac{1}{n}\left(\underline{X_1} + \cdots + \underline{X_n}\right), \frac{1}{n}\left(\overline{X_1} + \cdots + \overline{X_n}\right)\right], [\underline{U},\overline{U}]\right) \to 0(for\ n \to \infty).$$

If $\lambda < 0, \lambda X = [\lambda\overline{X},\lambda\underline{X}]$. We have

$$h([\frac{1}{n}(\lambda\overline{X_1} + \cdots + \lambda\overline{X_n}), \frac{1}{n}(\lambda\underline{X_1} + \cdots + \lambda\underline{X_n})], [\lambda\overline{U},\lambda\underline{U}]$$

$$= h\left(\left[\frac{1}{n}(-\lambda)\left(\underline{X_1} + \cdots + \underline{X_n}\right), \frac{1}{n}(-\lambda)\left(\overline{X_1} + \cdots + \overline{X_n}\right)\right], (-\lambda)[\underline{U},\overline{U}]\right)$$

$$= (-\lambda)h([\frac{1}{n}(\underline{X_1} + \cdots + \underline{X_n}), \frac{1}{n}(\overline{X_1} + \cdots + \overline{X_n})], [\underline{U},U]) \to 0(n \to \infty)$$

We have shown that when $X \in C_I(1)$, $\lambda X \in C_I(1)$.

Theorem 6: The space $C_I(1)$ is a QLS with the operations given by (3), (4) and the partial order relation given by (5).

Proof: For all $U, V, Z \in C_I(1)$ and $\lambda \in \mathbb{R}$ also, for $k \in \mathbb{N}$,

i) It is trivial that $U \preccurlyeq U$.

ii) When $U \preccurlyeq V$ and $V \preccurlyeq Z$ for all $1 \leq k < \infty$, we have $U_k \subseteq V_k$ and $V_k \subseteq Z_k$ due to (5). Since $U_k, V_k, Z_k \in P$ and P is a QLS, it becomes $U_k \subseteq Z_k$, for all $1 \leq k < \infty$. Because of (5), $U \preccurlyeq Z$.

iii) Let $U \preccurlyeq V$ and $V \preccurlyeq U$. In this case, for all $1 \leq k < \infty$, $U_k \subseteq V_k$ and $V_k \subseteq U_k$. Since $U_k, V_k \in P$, and P is a QLS, equality $U_k = V_k$ is obtained. So, $U = V$.

It can be easily seen that these two equations are satisfied:

iv) $U + V = V + U$,

v) $U + (V + Z) = (U + V) + Z$,

vi) There is $\theta = (\theta_P, \theta_P, \dots, \theta_P, \dots) \in C_I(1)$ to be provided $U + \theta = U$, where $\theta_P = 0$.

vii) It is very easy to see that

$$\lambda.(\beta.U) = \lambda.(\beta.U_1, \beta.U_2, \dots, \beta U_n, \dots)$$

$$= (\lambda\beta.U_1, \lambda\beta.U_2, \dots, \lambda\beta U_n, \dots) = (\lambda\beta).U,$$

viii) $\lambda(U+V) = \lambda(U_1 + V_1, U_2 + V_2, \dots, U_n + V_n, \dots)$
$= (\lambda U_1 + \lambda V_1, \lambda U_2 + \lambda V_2, \dots \lambda U_n + \lambda U_n, \dots)$
$= \lambda U + \lambda V,$

ix) $1.U = U,$

x) $0.U = 0.(U_1, U_2, \dots, U_n, \dots) = (0.U_1, 0.U_2, \dots, 0.U_n, \dots) = (0,0, \dots .0) = \theta,$

xi) $(\lambda+\beta).U = (\lambda+\beta).(U_1, U_2, \dots, U_n, \dots)$
$= ((\lambda+\beta).U_1, (\lambda+\beta).U_2, \dots, (\lambda+\beta).U_n, \dots),$

and it is obtained that

$$\lambda.U + \beta.U = \lambda.(U_1, U_2, \dots, U_n, \dots) + \beta.(U_1, U_2, \dots, U_n, \dots)$$

$$= (\lambda.U_1 + \beta.U_1, \lambda.U_2 + \beta.U_2, \dots, \lambda.U_n + \beta.U_n, \dots).$$

For all $1 \le k < \infty$, when $U \in C_I(1)$, since $U_k, V_k \in P$ and P is a QLS, we have

$$(\lambda+\beta).U_k \subseteq \lambda.U_k + \beta.U_k,$$

and thus, it is obtained that $(\lambda+\beta).U \subseteq \lambda.U + \beta.U$,

xii) If $U \preccurlyeq V$and $Z \preccurlyeq W$, for $1 \le k < \infty$, then $U_k \subseteq V_k$ and $Z_k \subseteq W_k$. It is founded that

$$U_k + Z_k \subseteq V_k + W_k, \text{ and } U + Z \preccurlyeq V + W.$$

xiii) If $U \preccurlyeq V$, since $U_k \subseteq V_k$, for $\lambda \in \mathbb{R}$, we get $\lambda.U_k \subseteq \lambda.V_k$. From here $\lambda.U \preccurlyeq \lambda.V$ obtained.
So, $C_I(1)$ is a quasilinear space with operations (3), (4) and (5).

Theorem 7: The space $C_I(1)$ is a normed quasilinear space with the function n defined as

$$n: C_I(1) \to \mathbb{R},\ n(U) = \sup_k \|U_k\|_P,$$

where $\|U_k\|_P = max\{|\underline{U}_k|, |\overline{U}_k|\}$.

Proof: For all $U, V \in C_I(1)$ and $\beta \epsilon \mathbb{R}$,

i) It is clear that $n(U) \ge 0$.

ii) $n(U) = 0 \Leftrightarrow \sup_k \|U_k\|_P = 0, (k \in \mathbb{N})$

$\Leftrightarrow$For all $k \in \mathbb{N}$, $\|U_k\|_P = 0$,

$\Leftrightarrow U = \theta$.

iii) $n(U+V) = \sup\limits_k \|U_k + V_k\|_P, \quad (k \in \mathbb{N})$

$\leq \sup\limits_k \{\|U_k\|_P + \|V_k\|_P\}$,

$= \sup\limits_k \{\|U_k\|_P\} + \sup\limits_k \{\|V_k\|_P\}$

$= n(U) + n(V)$,

iv) $n(\beta U) = \sup\limits_k \|\beta U_k\|_P, \quad (k \in \mathbb{N})$

$= \sup\limits_k \{|\beta| \|U_k\|_P\}, \quad (k \in \mathbb{N})$,

$= |\beta| . \sup\limits_k \{\|U_k\|_P\}, \quad (k \in \mathbb{N})$

$= |\beta| n(U)$.

v) Let us assume that $U \preccurlyeq V$. This implies it to be $U_k \subseteq V_k$, and $\|U_k\|_P \leq \|V_k\|_P$ for each $k \in \mathbb{N}$, U_k, $V_k \in P$, since P is a normed QLS. From here it is obtained $\sup\limits_k \|U_k\|_P \leq \sup\limits_k \|V_k\|_P$, and it means $n(U) \leq n(V)$.

vi) Let $\delta > 0$ be given and let $U, V \in C_I(1)$. Assume that there exists an element $U_\delta \in C_I(1)$ such that $U \preccurlyeq V + U_\delta$ and $n(U_\delta) \leq \delta$. To verify the last case, we must show that $U \preccurlyeq V$ in these conditions. From the hypothesis, we get $U_k \subseteq V_k + U_{\delta_k}$ and $\sup\limits_k \left\|U_{\delta_k}\right\|_P \leq \delta$ for each positive integer k. Since P is a normed QLS, we can say $U_k \subseteq V_k$ for each positive integer k. Hence, this means $U \preccurlyeq V$. This completes the proof.

4. Conclusion

We define an interval valued space and then present some topological characteristics and inclusion relations of this space. By demonstrating that, this sequence space has the quasilinear space structure described by Aseev, we also made a contribution to the study of quasilinear spaces. The paper serves as a guide for future research in a related field.

Acknowledgments

This article has been created from the second author's master's thesis, which was overseen by the first author.

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